Asymptoticity and non-expansivity

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Theorem (Schwartzman, 1952)

Let X be an infinite compact space and $T : X \to X$ a homeo. Then $\forall \epsilon > 0$, there are $x \neq y \in X$,

$$\sup_{n>0} \mathrm{dist}(T^n x, T^n y) < \epsilon.$$

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Corollary

- *Any infinite $\mathbb{Z}$-subshift $X \subset A^{\mathbb{Z}}$ admits asymptotic pair, i. e.*

$$\exists x, y \in X, \quad x_0 \neq y_0, x_n = y_n \quad \forall n \in \mathbb{N}^*.$$

- *Morse-Hedlund theorem (1940):*

  $p_X(n) \leq n$ *for some* n, *then the subshift* X *is finite.*

Higher rank action $\mathbb{Z}^d$, $d \geq 1$

Theorem (Boyle-Lind, 1997)

Let $\mathbb{Z}^d \overset{T}{\curvearrowright} X$ *be an infinite topological dynamical system. Set* $ND(X) =$

$$\{v \in \mathbb{S}^{d-1} : \forall \epsilon > 0, \exists x \neq y \in X \sup_{\langle g, v \rangle < 0} \mathrm{dist}(T_g x, T_g y) < \epsilon\}.$$

Then $ND(X)$ *is a closed non-empty set of* $\mathbb{S}^{d-1}$,

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Theorem (Boyle-Lind, Hochman '09)

For any closed set $N \subset \mathbb{S}^{d-1}$, *there is a* $\mathbb{Z}^d$ *subshift* X *such that*

$$N \cup (-N) = ND(X) \cup (-ND(X)).$$

Theorem (Robinson Crusoe Theorem (Donoso-Maass-P.))

Let G be an infinite countable group, X compact metric space and a continuous action $G \overset{T}{\curvearrowright} X$.
Let $O \subsetneq X$ be open, not closed and G-invariant

$$T_g(O) = O, \forall g \in G.$$

Then, for any neighborhood U of the boundary of O, $\exists$ a horoball H of G such that

$$O \cap \bigcap_{g \in H} T_g^{-1}(U) \neq \emptyset.$$

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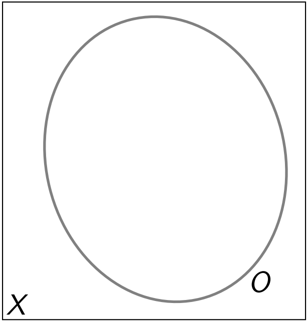
E.g.
$G = \mathbb{Z}$



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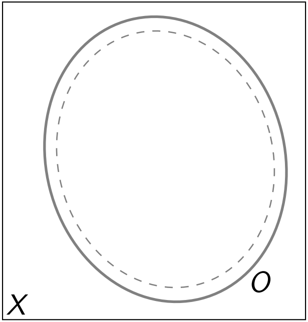

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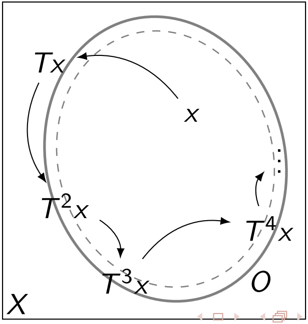

What is . . . a horoball?

G a countable topological group with a distance $d\colon G \times G \to \mathbb{R}_+$

- right invariant, *i.e.*, $d(gf, hf) = d(g, h), \quad \forall g, h, f \in G$;
- and proper, *i.e.*, every closed ball is compact.

E.g. G generated by a finite set $\mathcal{S}$

$$d(g, h) = \inf\{n \in \mathbb{N} : g^{-1}h = s_1 \cdots s_n \text{ for } s_1, \ldots, s_n \in \mathcal{S} \cup \mathcal{S}^{-1}\}$$

Busemann cocycle

$$\begin{array}{rcl} b\colon G & \hookrightarrow & C(G) \\ g & \mapsto & \left(b_g : x \mapsto d(g,x) - d(g,1_G)\right) \end{array}$$

$b(G)$ is relatively compact (Arzelà-Ascoli's theorem)

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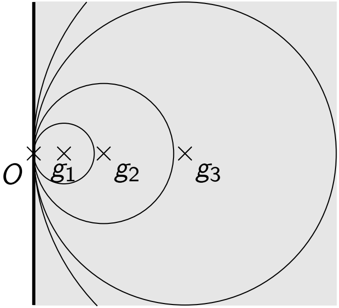
Border of G

$$\partial G = \overline{b(G)} \setminus b(G)$$

Horofunction $j \in \partial G$
Horoball $\{g \in G : j(g) < 0\}$ for $j \in \partial G$



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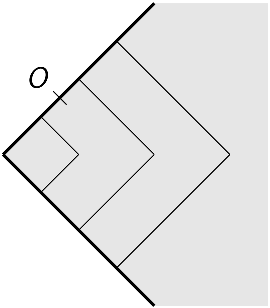

Corollary (DMP)

Let $G \overset{T}{\curvearrowright} X$ and $G \overset{S}{\curvearrowright} Y$ be topological dyn. syst. with G infinite countable, and let $\pi : X \to Y$ be a factor map.
Then, either

- *π is bounded to one,*
- *or $\forall \epsilon > 0$ there exist a horoball $H \in \partial G$ and $x \neq y \in X$, s.t. $\pi(x) = \pi(y)$ and*

$$\sup_{g \in H} \mathrm{dist}(T_g x, T_g y) < \epsilon.$$

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If Y is the trivial factor and X is infinite, there are $x \neq y \in X$,

$$\sup_{g \in H} \mathrm{dist}(T_g x, T_g y) < \epsilon.$$

- $G = \mathbb{Z}$ — Sol Schwartzman (52)
- $G = \mathbb{Z}^d$ — Boyle-Lind (97)

H is said non-deterministic, ∂H is said non-expansive.

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Proof :

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Proof : Apply R. Crusoe theorem to

$$X_\pi^2 = \{(x, y) \in X^2 : \pi(x) = \pi(y)\}, \qquad G \overset{T \times T}{\curvearrowright} X_\pi^2$$
$$O = X_\pi^2 \setminus \{(x, x) : x \in X\},$$
$$U = \{(x, y) \in X_\pi^2 : \mathrm{dist}(x, y) < \epsilon\}.$$

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Yes

for some groups

Directed version for G

Theorem

Let $G \overset{T}{\curvearrowright} X$ and $G \overset{S}{\curvearrowright} Y$ be topological dyn. syst. with G infinite countable with a proper bi-invariant distance.
Let $\pi : X \to Y$ be a factor map that is not bounded to one.
Let $k \in G$ be an element of infinite order.
Then, for any $\epsilon > 0$, there exists a horoball H s.t.

- $\exists x \neq y$ *with* $\pi(x) = \pi(y)$ *s.t.* $\sup_{g \in H} \mathrm{dist}(T_g x, T_g y) < \epsilon$
- *The horoball H does not contain k.*

Proof given by a directed Robinson Crusoe theorem.

Directed version for $\mathbb{Z}^d$

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Then,

- *the origin belongs to the convex hull generated by the elements in* $ND(X)$*,*
- *or equivalently* $\bigcap_{v\in ND(X)}\{w \in \mathbb{R}^d : \langle w, v\rangle < 0\} = \emptyset$*,*
- *or equivalently,* $\forall w \in \mathbb{R}^d$ $ND(X) \cap \{x \in \mathbb{R}^d : \langle x, w\rangle \leq 0\} \neq \emptyset$*.*

First claimed by Guillon-Kari-Zinoviadis for $d = 2$.

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This generalizes Schwartzman's result

Problems: Group geometric flavour

- Find a good metric on G to have big horoballs.
- Find good metric on G so that the set of non-deterministic horoballs is closed.
- Version of Robinson Crusoe for connected group like $\mathbb{R}^d, \mathrm{SL}(\mathrm{d}, \mathbb{R}), \ldots$ (tilings)

Problems: Symbolic flavour

For any norm $\|\cdot\|$, any closed half space $H = \{x \in \mathbb{R}^d : \langle x, w \rangle \leq 0\}$

- Define *H*-special pattern, (similar to special word):

$$\forall n \geq 0, \exists \mathrm{P}_1 \neq \mathrm{P}_2 \in \mathcal{L}_{B_{n+1}(O) \cap H}(X), \text{ s.t. } \mathrm{P}_1|_{B_n(O)} = \mathrm{P}_2|_{B_n(O)}.$$

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- Define a Morse-Hedlund theorem on $\mathbb{Z}^d$, G

$$|\mathcal{L}_{B_{n+1}(O)\cap H}(X)| = |\mathcal{L}_{B_n(O)\cap H}(X)| \text{ for some } n \implies X \text{ finite.}$$

Link with complexity?

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- Nivat conjecture version for $\mathbb{Z}^d$?

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- For any closed set $N \subset \mathbb{S}^{d-1}$, s.t. $O \in \mathrm{conv}(\mathrm{N})$, is it true that $N = ND(X)$ for some $\mathbb{Z}^d$-system?
  What's happen if one restricts the dynamics (SFT, aperiodic, cut-and-project, minimal, . . .) ?

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  What's happen if one restricts the dynamics (SFT, aperiodic, cut-and-project, minimal, ...) ?
- Is it true that any $\mathbb{Z}^d$-SFT is a product of finitely many $\mathbb{Z}^d$-SFT, that are unfactorizable?

Problems: Dynamical flavour

- If a homeomorphism T is minimal and expansive, then the space is totally disconnected — Mañé '79
  Version for $\mathbb{Z}^2$ action?
- Any homeomorphism of positive entropy admits an asymptotic pair. — Blanchard-Host-Ruette '02
  Version for $\mathbb{Z}^d$?