Problèmes de domino et entropie

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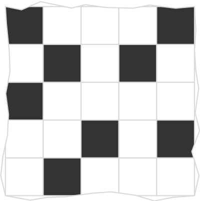
Premières journées IZES

SFT, domino, (a)periodicity

Subshifts

- $\mathcal{A}$ a finite **alphabet** (□, ■, sometimes ■);
- $\mathbb{Z}^d$ a d-dimensional **grid**;
- $\mathcal{A}^*$ finite **patterns**;
- $\mathcal{A}^{\mathbb{Z}^d}$ infinite **configurations** (or **tilings**).

$d = 2$, $\mathcal{F} = \{$■■, $\begin{smallmatrix}■\\■\end{smallmatrix}\}$:



Subshift $X_{\mathcal{F}}$: the set of configurations without any **forbidden pattern** from a set $\mathcal{F} \subset \mathcal{A}^*$.

A pattern or configuration with no forbidden pattern is **admissible**.

Domino problem

Domino

Input A SFT (= alphabet + set of forbidden patterns)

Output Is there an admissible configuration ?

Intuition

Equivalent to the existence of arbitrarily large admissible patterns.

$$D(X_{\mathcal{F}}) \Leftrightarrow \forall n, \exists \text{admissible pattern of size } n$$

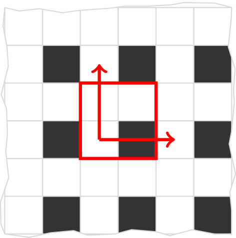
Naturally Π_1^0-computable (= co-recursively enumerable)

Periodicity

A configuration x **has period** $\vec{v} \in \mathbb{Z}^2$ si

$$x_{\vec{i}+\vec{v}} = x_{\vec{i}} \text{ for all } i.$$

It is k**-periodic** if it admits k independent periods (and no more).



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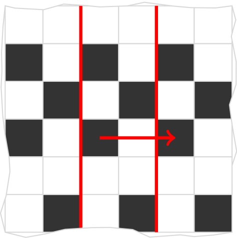

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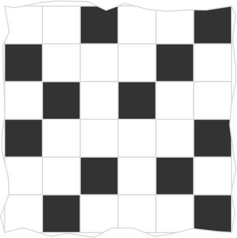

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strongly periodic $:=$ d-periodic; (strongly) aperiodic $:=$ 0-periodic

Domino problem

Domino

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Output Is there an admissible configuration?

This problem is decidable for SFT that contain a strongly periodic configuration.

Algorithm

For n going from 1 to ∞:

- List all patterns of size n.
- No admissible pattern: answer **No**.
- A d-periodic admissible pattern: answer **Yes**.
- Otherwise: continue.

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Dimension ≥ 2

The algorithm fails, and Domino is in fact Π_1^0-complete (Berger 66).

Embedding of universal computation in aperiodic SFT.

(A)periodic domino

Periodic domino problem

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Intuition

We look for a *d*-periodic admissible pattern:

$$DP(X_{\mathcal{F}}) \Leftrightarrow \exists n, \exists \text{"periodic" admissible pattern of size } n$$

Naturally Σ_1^0-computable (recursively enumerable).

in fact: it is Σ_1^0-complete in dimension ≥ 2.

Aperiodic domino problem

Aperiodic domino

Input A SFT (= alphabet + set of forbidden patterns)

Output Is there a **strongly aperiodic** admissible configuration?

Aperiodic domino problem

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Intuition

$$DA(X_{\mathcal{F}}) \Leftrightarrow \exists \text{an admissible and aperiodic configuration}$$

...quantifying on infinite objects.

Naturally Σ_1^1-computable (analytical).

...could it be arithmetic?

Aperiodic domino problem

Aperiodic domino

Input: A SFT (= alphabet + set of forbidden patterns)

Output: Is there a **strongly aperiodic** admissible configuration?

Theorem (Grandjean, H., Vanier)

The aperiodic domino is Π_1^0(co-recursively enumerable)-complete in dimension 2.

Theorem (Callard, H.)

The aperiodic domino is Σ_1^1-complete in dimension ≥ 4.

Arithmetisation

In dimension 2,

Shepherd's lemma

A SFT has a strongly aperiodic configuration.

$$\Updownarrow$$

For all n, there is an admissible $f(n) \times f(n)$ pattern that breaks every period of length $\leq n$ in a "concentric" manner.

Conclusion

Complexity jump in the aperiodic domino:

d	SFT	sofic	effectif
2	Π^0_1	Π^0_1	Π^0_1
3	**?**	Σ^1_1	Σ^1_1
≥ 4	Σ^1_1	Σ^1_1	Σ^1_1

- Σ^1_1-complete in general: unexpected.
- Complexity drops in dimension 2 for geometric reasons: unexpected.
- Conjecture: complexity drops in dimension 3 for other (SFT-specific) reasons — ongoing work with Callard & Salo
- Huge complexity jump SFT / sofic? — would be unexpected.

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Open question (Callard, H. @Dyadisc)

Can we get any money from Fabien Durand's ANR with this stuff?

Domino with complexity promises

Domino with promises

Entropy

$$h(X) = \lim \frac{\log \#\mathcal{L}_n(X)}{n^d}$$

Domino with a positive entropy promise

Input A SFT that is empty or with positive entropy

Output Is there an admissible configuration?

This version, and the version with zero entropy, are equivalent to the domino.

Domino with promises

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In dimension 2,

Theorem (Kari, Moutot)

The domino problem with a promise that $\mathcal{L}_n(X) \leq n^2$ for some n is decidable.
The domino problem with a promise that $\mathcal{L}_n(X) \leq (1+\varepsilon)n^2$ is undecidable.

Aperiodic domino with a promise

Aperiodic domino with positive entropy promise

Input A SFT that is empty or with positive entropy

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Aperiodic domino with a promise

Aperiodic domino with positive entropy promise

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Theorem (Callard, H.)

In dimension d, if

$$\limsup_n \frac{\log \#\mathcal{L}_n(X)}{n^k} = \infty,$$

then X contains a point that is not $(d-k)$-periodic.

This is equivalent to the domino: much easier than the aperiodic domino.

The version with a zero entropy promise is equivalent to the aperiodic domino.

Domino inside a subshift

X-domino

Let X be a subshift.

X-domino

Input A SFT Y

Output Is there a configuration in $X \cap Y$?

X-sofic domino

Input A sofic shift Y

Output Is there a configuration in $X \cap Y$?

In the zero entropy world

Minimal case

If X is minimal and decidable, then the X-domino is decidable.

In the zero entropy world

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In dimension 1,

Theorem (Salo)

The X-sofic domino problem for X a substitutive shift is decidable.
If X is a quasiminimal subshift, then X-sofic domino can be undecidable.

(Thanks to Marie-Pierre Béal)

Ongoing work with Pierre Béaur

X-sofic domino is decidable when X is any s-adic shift.
(the sequence of substitutions can be fixed, arbitrary or regular)

In the nonzero entropy world

"The" undecidability proof

In dimension ≥ 2, if X factors onto a full shift, then X-domino is undecidable.

Theorem (Desai, extending Johnson & Madden)

If X has positive entropy and is corner-gluing, then it factors onto a full shift.

There are counterexamples when removing the corner-gluing assumption (Boyle & Schraudner).

Some naive questions

- In dimension 1, are there decidable subshifts where the X-domino is undecidable?

Kari's snakes, 2003

The X-sofic domino is undecidable where X is the 1d subshift on alphabet $\{a, b, a^{-1}, b^{-1}\}$ defined by:

$$\mathcal{F} = \{w : |w|_a = |w|_{a^{-1}} \text{ and } |w|_b = |w|_{b^{-1}}\}.$$

(Thanks to Nicolás Bitar)

- Can we find general methods that do not depend on a substitutive structure?
- In dimension 2, any decidability results for interesting classes of (non-minimal) subshifts? for some positive entropy subshifts?