Recent developments in finite alphabet rank $\mathcal{S}$-adic sequences

Rencontre IZES

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2. Finite alphabet rank $\mathcal{S}$-adic subshifts
3. Recent developments
   - Factors
   - Asymptotic pairs
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Definitions: general theory

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4. We will consider only **minimal** subshifts X, *i.e.* s.t. every orbit $\mathrm{orb}_S(x) \coloneqq \{S^k x : k \in \mathbb{Z}\}$ is dense in X.

Definitions: $\mathcal{S}$-adic subshifts

1. For $n \geq 0$, let $\tau_n \colon \mathcal{A}_{n+1} \to \mathcal{A}_n^+$ be a map, which we call **substitution**.
    - We define the extension $\tau_n \colon \mathcal{A}_{n+1}^+ \to \mathcal{A}_n^+$ by concatenation:
    $$\tau_n(a_1 a_2 \dots a_\ell) = \tau_n(a_1)\tau_n(a_2)\dots\tau_n(a_\ell)$$
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2. Then, $\boldsymbol{\tau} = (\tau_n)_{n\geq 0}$ is called an **$\boldsymbol{\mathcal{S}}$-adic sequence** if

$$\lim_{n\to+\infty} \min_{a\in\mathcal{A}_n} |\tau_{[0,n)}(a)| = +\infty.$$

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$$\lim_{n\to+\infty} \min_{a\in\mathcal{A}_n} |\tau_{[0,n)}(a)| = +\infty.$$
3. We define the ***n*-th $\mathcal{S}$-adic subshift** $X_{\boldsymbol{\tau}}^{(n)}$ as follows:
   - $x \in X_{\boldsymbol{\tau}}^{(n)}$ iff $\forall \ell \geq 0$, $\exists N > n, a \in \mathcal{A}_N$ s.t. $x_{[-\ell,\ell]}$ occurs in $\tau_{[n,N)}(a)$.

We set $X_{\boldsymbol{\tau}} := X_{\boldsymbol{\tau}}^{(0)}$.

Definitions: factorizations

1. **Fact.** [1] For any $n > 0$ and $x \in X_{\tau}$, there exists a $\boldsymbol{\tau_{[0,n)}}$**-factorization** of x in $X_{\tau}^{(n)}$, that is, $k \in \mathbb{Z}$ and $y \in X_{\tau}^{(n)}$ s.t.

$$x = S^k \tau_{[0,n)}(y) \text{ and } 0 \leq k < |\tau_{[0,n)}(y_0)|.$$

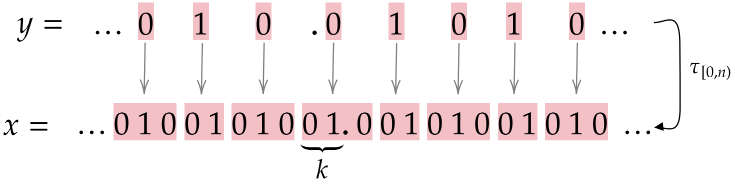

Figure: Here, $\tau_{[0,n)} \colon \{0,1\} \to \{0,1\}^+$ maps $0 \mapsto 010$ and $1 \mapsto 01$.

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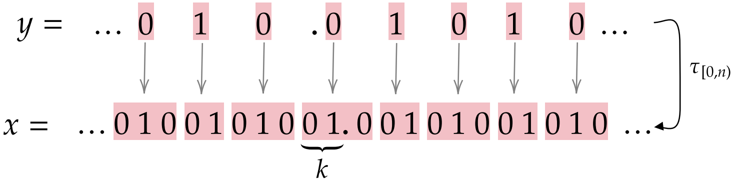

Figure: Here, $\tau_{[0,n)} : \{0,1\} \to \{0,1\}^+$ maps $0 \mapsto 010$ and $1 \mapsto 01$.

2. $\boldsymbol{\tau}$ is **recognizable** if (k, y) is unique for all $n > 0$ and $x \in X_{\boldsymbol{\tau}}$.

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Definitions: alphabet rank

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2. A **contraction** of $\boldsymbol{\tau}$ is an $\mathcal{S}$-adic sequence of the form $\boldsymbol{\tau}' = (\tau_{[0,n_1)}, \tau_{[n_1,n_2)}, \dots)$.

3. $\boldsymbol{\tau}$ and $\boldsymbol{\tau}'$ generate the same $\mathcal{S}$-adic subshift $X_{\boldsymbol{\tau}}$.

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4. If $\text{Alph-Rank}(\boldsymbol{\tau}) < \infty$, then there is a contraction $\boldsymbol{\tau}' = (\tau_{[n_k,n_{k+1})})_{k\geq 0}$ s.t. $\#\mathcal{A}_{n_k} = \text{Alph-Rank}(\boldsymbol{\tau})$, $\forall k \geq 1$.

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The topological rank

1. **Framework**: minimal subshifts $X_{\boldsymbol{\tau}}$, where $\boldsymbol{\tau}$ has finite alphabet rank.

[2] Corollary 1.4 in *Symbolic factors of $\mathcal{S}$-adic subshifts of finite alphabet rank, B. Espinoza, 2022.*

The topological rank

1. **Framework**: minimal subshifts $X_{\boldsymbol{\tau}}$, where $\boldsymbol{\tau}$ has finite alphabet rank.
2. $\tau \colon \mathcal{A} \to \mathcal{B}^+$ is **proper** if $\exists a, b \in \mathcal{B}$ s.t. $\forall c \in \mathcal{A}$, $\tau(c)$ starts with a and ends with b.
3. $\boldsymbol{\tau} = (\tau_n)_{n \geq 0}$ is proper if every τ_n is proper.

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3. $\boldsymbol{\tau} = (\tau_n)_{n \geq 0}$ is proper if every τ_n is proper.
4. For X a minimal subshift, the following are equivalent[2]:
   - $X = X_{\boldsymbol{\tau}}$ for some $\boldsymbol{\tau}$ s.t. Alph-Rank$(\boldsymbol{\tau}) < +\infty$.
   - $X = X_{\boldsymbol{\tau}}$ for some recognizable and proper $\boldsymbol{\tau}$ s.t. Alph-Rank$(\boldsymbol{\tau}) < +\infty$.

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*A minimal subshift X has **topological rank K** if K is the least number for which $X = X_{\boldsymbol{\tau}}$ for some recognizable and proper $\boldsymbol{\tau}$ s.t. Alph-Rank$(\boldsymbol{\tau}) = K$.*

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5. **Framework**: it is equivalent to work with finite top. rank subshifts.

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Examples of finite top. rank subshifts

1. substitutive and linearly recurrent subshifts;
2. Sturmian, (natural codings of) minimal IET, dendric subshifts;
3. some Toeplitz subshifts;
4. subshifts X whose word-complexity function p_X has linear or non-superlinear growth;
5. for any $f : \mathbb{N} \to \mathbb{N}$ s.t. $\frac{1}{n} \log f(n) \to 0$ there exists X with finite top. rank and $p_X(n)/f(n) \to +\infty$.

Known properties of finite top. rank subshifts

Although the finite top. rank class contains a broad spectrum of minimal subshifts, it is possible to prove general theorems for it.

Known properties of finite top. rank subshifts

Although the finite top. rank class contains a broad spectrum of minimal subshifts, it is possible to prove general theorems for it. In a finite top.

rank subshift $X_{\boldsymbol{\tau}}$:

1. the topological entropy is zero (Handelman-Boyle, '94);
2. the are only finitely many ergodic measures, and they can be computed from $\boldsymbol{\tau}$ (Bezuglyi *et. al.* '13);
3. the dimension group can be computed from $\boldsymbol{\tau}$ (Herman *et. al.*, '99);
4. there is criteria stated in terms of $\boldsymbol{\tau}$ for deciding if a given complex number is an eigenvalue (Durand *et. al.*, '19);
5. among other properties and tools.

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Factors

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   - this is an equicontinuous system.
3. Example: (X, S) is a system.
   - this is an expansive system.
4. A **factor map** of X is a continuous map $\pi\colon X \to Y$ s.t. $T\pi(x) = \pi(Sx)$, $\forall x \in X$. Then, Y is a **factor** of X.

5. Factors are (topological) representations of X in Y.

Subshift factors

1. Suppose that Y is a subshift.

2. (Curtis–Hedlund–Lyndon theorem) There exists $r \geq 0$ and $\phi\colon \mathcal{A}^{2r+1} \to \mathcal{B}$ s.t. π is described by

$$x = (x_n)_{n \in \mathbb{Z}} \mapsto \pi(x) = (\phi(x_{[n-r,n+r]}))_{n \in \mathbb{Z}}.$$

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3. We first consider subshift factors of $\mathcal{S}$-adic subshifts.

4. **Our framework**: a minimal subshift $X = X_{\boldsymbol{\tau}}$ s.t. $\boldsymbol{\tau} = (\tau_n\colon \mathcal{A}_{n+1} \to \mathcal{A}_n^+)_{n\geq 0}$ is recognizable, proper, and of finite alphabet rank.

Subshift factors: $\mathcal{S}$-representation

Proposition (First representation of π)

Let $\pi\colon X_{\boldsymbol{\tau}} \to Y$ be a subshift factor, with $Y \subseteq \mathcal{B}^{\mathbb{Z}}$. Then, there exist $\ell \geq 0$ and $\sigma\colon \mathcal{A}_\ell \to \mathcal{B}^+$ s.t.

1. *If $x \in X_{\boldsymbol{\tau}}$ and (k, y) is its $\tau_{[0,\ell)}$-factorization in $X_{\boldsymbol{\tau}}^{(\ell)}$, then $\pi(x) = S^k\sigma(y)$.*
2. *$|\sigma(a)| = |\tau_{[0,\ell)}(a)|$ for all $a \in \mathcal{A}_\ell$.*

Conversely, if σ and ℓ satisfy (2) then (1) defines a factor map π.

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2. Note that Y is generated by $\boldsymbol{\sigma}' = (\sigma, \tau_\ell, \tau_{\ell+1}, \dots)$.
3. $\boldsymbol{\sigma}'$ is proper and Alph-Rank$(\boldsymbol{\sigma}')$ = Alph-Rank$(\boldsymbol{\tau})$.
4. Problem: $\boldsymbol{\sigma}'$ is recognizable iff π is a conjugacy.

Subshift factors: $\mathcal{S}$-representation

Theorem (E. 2021)

Up to a contraction of $\boldsymbol{\tau}$, there exist an $\mathcal{S}$-adic sequence $\boldsymbol{\sigma} = (\sigma_n \colon \mathcal{B}_{n+1} \to \mathcal{B}_n^+)_{n\geq 0}$ and morphisms $\phi_n \colon \mathcal{A}_n \to \mathcal{B}_n^+$ s.t.

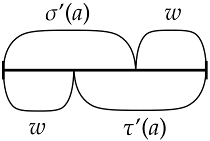
1. *$Y = X_{\boldsymbol{\sigma}}$, $\boldsymbol{\sigma}$ is proper, **recognizable**, Alph-Rank$(\boldsymbol{\sigma}) \leq$ Alph-Rank$(\boldsymbol{\tau})$.*
2. *For $n \geq \ell$,*

$$\begin{array}{ccc} X_{\boldsymbol{\tau}}^{(\ell)} & \xrightarrow{\ \phi_\ell\ } & X_{\boldsymbol{\sigma}}^{(\ell)} \\ {\scriptstyle \tau_{[0,\ell)}}\downarrow & \searrow^{\sigma} & \downarrow{\scriptstyle \sigma_{[0,\ell)}} \\ X_{\boldsymbol{\tau}} & \xrightarrow[\ \pi\ ]{} & Y \end{array} \qquad\qquad \begin{array}{ccc} X_{\boldsymbol{\tau}}^{(n+1)} & \xrightarrow{\ \phi_{n+1}\ } & X_{\boldsymbol{\sigma}}^{(n+1)} \\ {\scriptstyle \tau_n}\downarrow & & \downarrow{\scriptstyle \sigma_n} \\ X_{\boldsymbol{\tau}}^{(n)} & \xrightarrow[\ \phi_n\ ]{} & X_{\boldsymbol{\sigma}}^{(n)} \end{array}$$

1. Corollary: the top. rank of Y is at most the one of $X_{\boldsymbol{\tau}}$.
2. The proof of the theorem is constructive and allows to compute $\boldsymbol{\sigma}$ and ϕ_n.

Subshift factors: application to substitutions

1. $\sigma', \tau' \colon \mathcal{A} \to \mathcal{B}^+$ are **rotationally conjugate** if there exists a word w s.t. $\sigma(a)w = w\tau(a)$ for all $a \in \mathcal{A}$.



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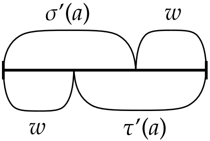

Proposition

Let $\tau \colon \mathcal{A} \to \mathcal{A}^+$, $\sigma \colon \mathcal{B} \to \mathcal{B}^+$ be primitive and proper substitutions. There exists a computable constant $n = n(\tau, \sigma)$ s.t. X_σ is a factor of X_τ iff there exist $\phi \colon \mathcal{A} \to \mathcal{B}^+$ and $p, q, r \leq n$ s.t.

1. *$\sigma^p \circ \phi$ and $\phi \circ \tau^q$ are rotationally conjugate;*
2. *$|\tau^n(a)| = |\sigma^r \phi(a)|$ for all $a \in \mathcal{A}$.*

Subshift factors: substitutive case

1. (Durand-Leroy '18) There is an algorithm that decides whether X_σ is a factor of X_τ.

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Question

Use this approach to continue exploring subshift factors of substitutive subshifts or other classes with finite top. rank.

More on subshift factors

Suppose that the minimal subshift X has finite topological rank.

1. Any self factor $\pi\colon X \to X$ is invertible (*i.e.* X is coalescent).

2. If $\pi\colon X \to Y$ is an infinite subshift factor, then $\exists k \in \mathbb{N}$ s.t. $\#\pi^{-1}(y) = k$ for almost all $y \in Y$.

Theorem

The number of subshift factors of X up to conjugacy is finite.

Other types of factors

Suppose that X is a minimal subshift with finite top. rank.

1. **Proposition:** if $\pi\colon X \to Y$ is a factor and Y is totally disconnected, then Y is either a subshift or an odometer.

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Question

What can be said of factors $\pi \colon X \to Y$ s.t. Y is a ***connected*** *topological space?*

- This case correspond to a geometrical representation of X.
- It is oppposite to the totally disconnected case.
- It is a direction that has not been explored even in the substitutive case.

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Asymptotic pairs

1. A (centered) **asymptotic pair** in X is a tuple $(x, y) \in X \times X$ s.t. $x_{(-\infty,0)} = y_{(-\infty,0)}$ and $x_0 \neq y_0$.

Theorem (E., Maass, 2020)

Let $\boldsymbol{\tau} = (\tau_n \colon \mathcal{A}_{n+1} \to \mathcal{A}_n^+)_{n\geq 0}$ *be such that*

$$\textit{Alph-Rank}(\boldsymbol{\tau}) = \liminf_{n\to+\infty} \#\mathcal{A}_n < \infty.$$

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2. This extends a previous result by Donoso *et. al.* from 2016.
3. The theorem can be used to bound the number of *automorphisms* and *symbolic factors* of $X_{\boldsymbol{\tau}}$ in the minimal case.

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Other types of factors

Let X be a minimal subshift.

1. The **complexity function** $p_X\colon \mathbb{N} \to \mathbb{N}$ is defined by

$$p_X(n) = \text{N}^\circ \text{ of words of length } n \text{ occurring in at least one } x \in X.$$

2. X has **linear growth** if

$$\limsup_{n\to+\infty} p_X(n)/n < +\infty.$$

3. X has **non-superlinear growth** if

$$\liminf_{n\to+\infty} p_X(n)/n < +\infty.$$

Linear growth subshifts

- The **root** $\mathrm{root}(w)$ of a word w is the shortest word v such that $w = v^k$ for some $k \geq 1$.
- Example: $\mathrm{root}(ababab) = ab$ and $\mathrm{root}(abababababa) = abababababa$.

Theorem

A minimal subshift X has linear growth if and only if there exist $C \geq 1$ and an $\mathcal{S}$-adic sequence $\boldsymbol{\tau} = (\tau_n \colon \mathcal{A}_{n+1}^+ \to \mathcal{A}_n)_{n\geq 0}$ generating X such that for every $n \geq 1$:

1. $\mathrm{root}(\tau_{[0,n)}(\mathcal{A}_n)) \coloneqq \{\mathrm{root}(\tau_{[0,n)}(a)) : a \in \mathcal{A}_n\}$ *has at most C elements;*
2. $|\tau_{[0,n)}(a)| \leq C \cdot |\tau_{[0,n)}(b)|$ *for every $a, b \in \mathcal{A}_n$;*
3. $|\tau_n(a)| \leq C$ *for every $a \in \mathcal{A}_n$.*

Linear growth subshifts

1. The previous characterization allows to recover known results about linear growth subshifts.
2. Cassaigne's Theorem: $p_X(n+1) - p_X(n)$ is bounded.
3. X has finitely many ergodic measures.
4. X has finite topological rank.

Non-superlinear growth subshifts

Theorem

A minimal subshift X has non-superlinear growth if and only if there exist $C \geq 1$ and an $\mathcal{S}$-adic sequence $\boldsymbol{\tau} = (\tau_n \colon \mathcal{A}_{n+1}^+ \to \mathcal{A}_n)_{n \geq 0}$ generating X such that for every $n \geq 1$:

1. $\mathrm{root}(\tau_{[0,n)}(\mathcal{A}_n)) = \{\mathrm{root}(\tau_{[0,n)}(a)) : a \in \mathcal{A}_n\}$ *has at most C elements;*
2. $|\tau_{[0,n)}(a)| \leq C \cdot |\tau_{[0,n)}(b)|$ *for every $a, b \in \mathcal{A}_n$.*

Observations

1. The conditions depend on the unbounded product $\tau_{[0,n)}$, and thus are "non-local".

Question

Is it possible to obtain a "local" characterization of subshifts with linear growth?

2. "Local" characterizations have been obtained for other classes, such as Sturmian, IET, and dendric.

Recent developments in finite alphabet rank $\mathcal{S}$-adic sequences

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