LANGUAGES OF GENERAL INTERVAL EXCHANGES

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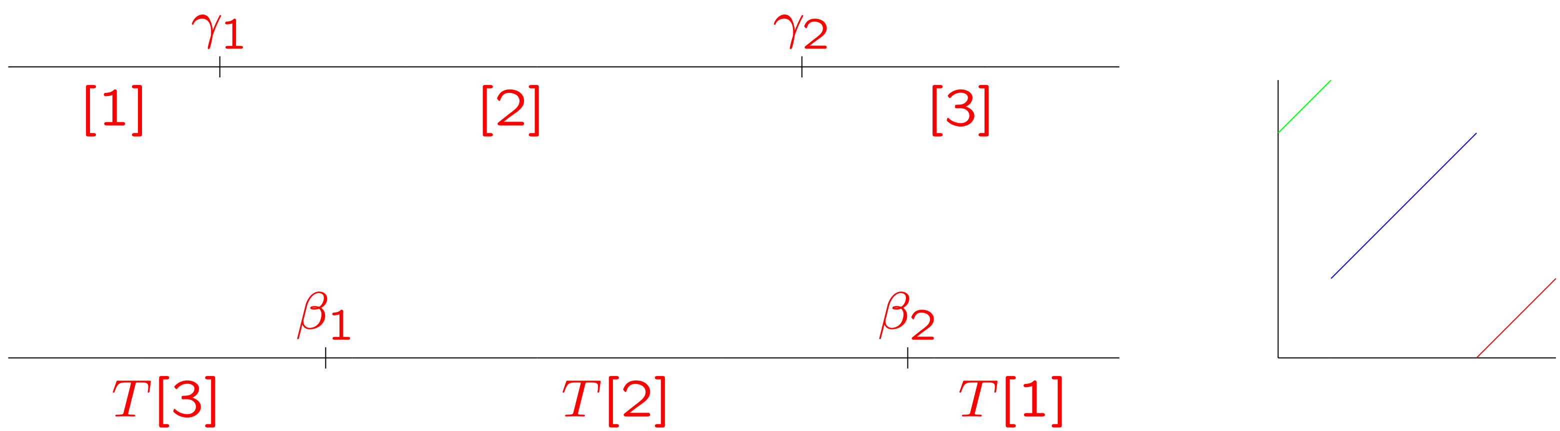
STANDARD INTERVAL EXCHANGES



Natural coding (for a given choice of γ_i, which determines their images β_j).

$\mathcal{A}$ finite alphabet, $x \in [0, 1[$,

$x \to (x_n) \in \mathcal{A}^{\mathbb{Z}}$ where $x_n = e$ if $T^n x$ falls into interval number e.

RAUZY's QUESTION

Characterize the languages generated by natural codings of iet.

Morse - Hedlund 1940 + Coven - Hedlund 1973 for irrational rotations (= standard 2-iet), defining Sturmian languages.

Ferenczi - Zamboni 2008 for i.d.o.c. standard iet.

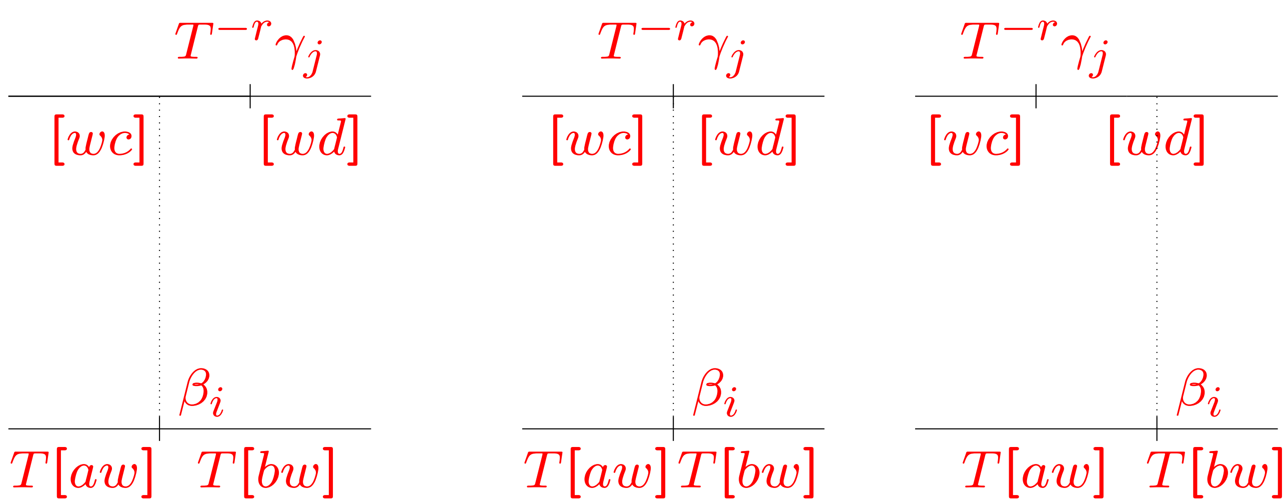
(Kanel) Belov - Chernyatev 2010 for standard iet and standard iet with flips. The minimal cases are also characterized.

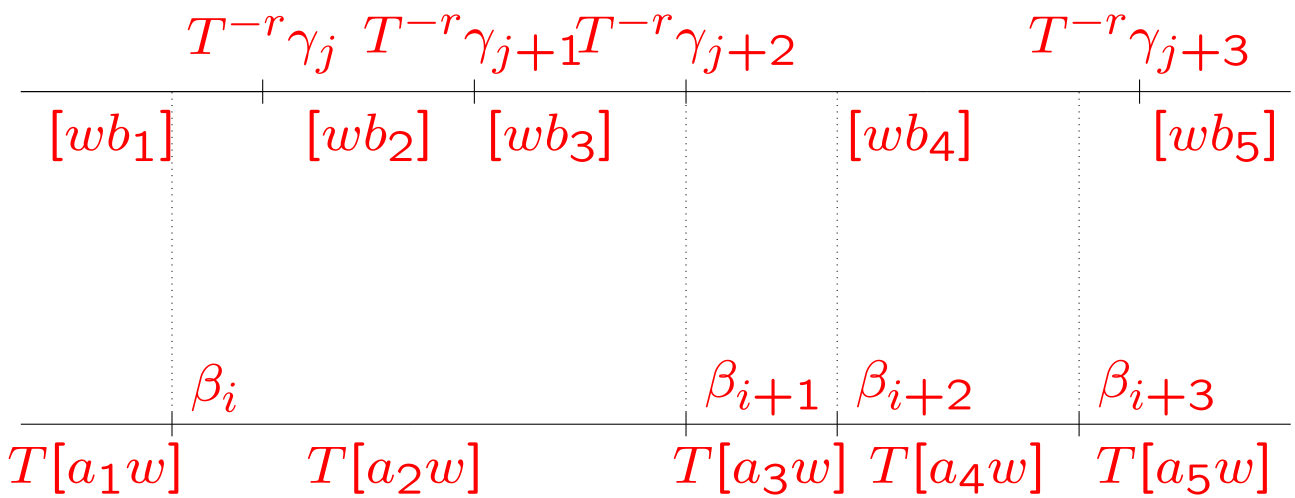
ORDER CONDITION

A language L satisfies an order condition if

there exist two (total) orders on $\mathcal{A}$, denoted by $<_A$ and $<_D$, such that whenever aw, bw, wc, wd are in L for a word w and letters $a \neq b$ and $c \neq d$ (w is called bispecial), then if $a <_A b$, $c <_D d$, either awd or bwc (or both) is not in L.

Every iet language satisfies an order condition




$T^{-r}\gamma_j$
$T^{-r}\gamma_{j+1}$
$T^{-r}\gamma_{j+2}$
$T^{-r}\gamma_{j+3}$
$[wb_1]$
$[wb_2]$
$[wb_3]$
$[wb_4]$
$[wb_5]$
β_i
β_{i+1}
β_{i+2}
β_{i+3}
$T[a_1w]$
$T[a_2w]$
$T[a_3w]$
$T[a_4w]$
$T[a_5w]$

DENDRIC LANGUAGES

For a bispecial w, let $A = \{a \in \mathcal{A}; aw \in L\}$, $D = \{b \in \mathcal{A}; wb \in L\}$

w is a weak bispecial if $\#\{awb \in L, a \in \mathcal{A}, b \in \mathcal{A}\} < \#A + \#D - 1$

w is a locally strong bispecial if there exist $A' \subset A$, $D' \subset D$, $\#\{awb \in L, a \in \mathcal{A}', b \in \mathcal{D}'\} > \#A' + \#D' - 1$.

A language is dendric <=> no weak bispecial and no locally strong bispecial.

The order condition implies (strictly) the absence of strong bispecials, but allows the presence of finitely many weak bispecials.

STANDARD CRITERION

Theorem (Belov?) L is the natural coding of a standard iet <=>
- order condition
- recurrence: for each $w \in L$ there exists a nonempty $v \in L$, such that wv ends with w.

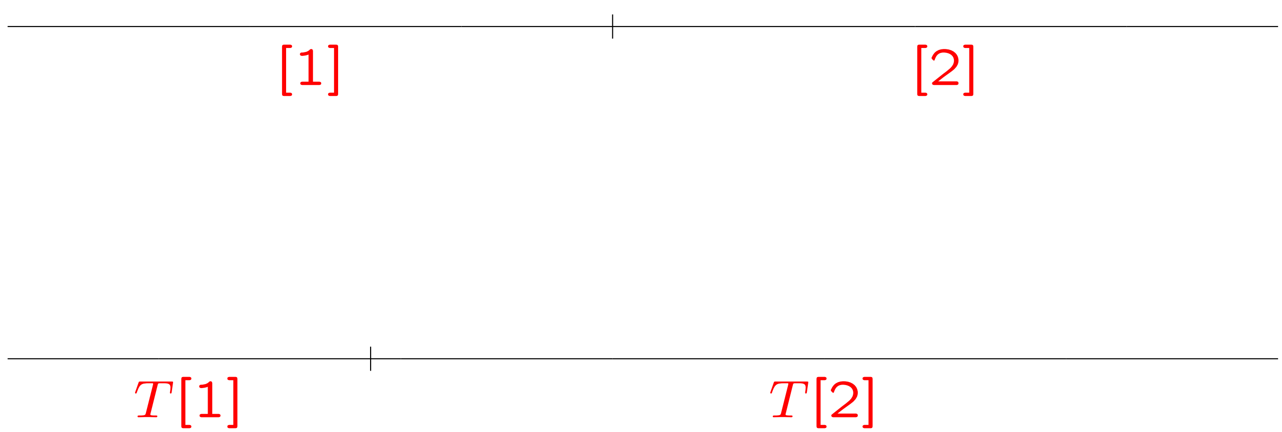
Proof
Order conditions plus recurrence imply L is a union of finitely many uniformly recurrent (possibly periodic) languages.
Then one can build an invariant measure positive on all cylinders.
Take the iet T with the same orders and I_i of length $\mu[i]$.
The natural coding of T has the same words of length 1, 2, 3 ...as L (Morse - Hedlund for 2 intervals).

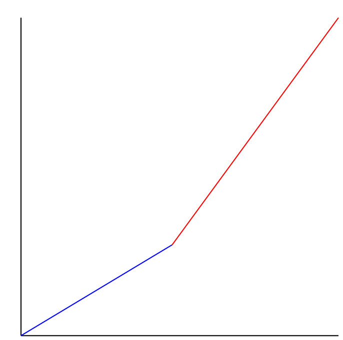
AFFINE IET

Is the order condition alone enough to make L the language of a standard iet? NO.

Counter-example Fake Sturmian: L generated by ...22221111....

BUT L is the natural coding of an affine iet.

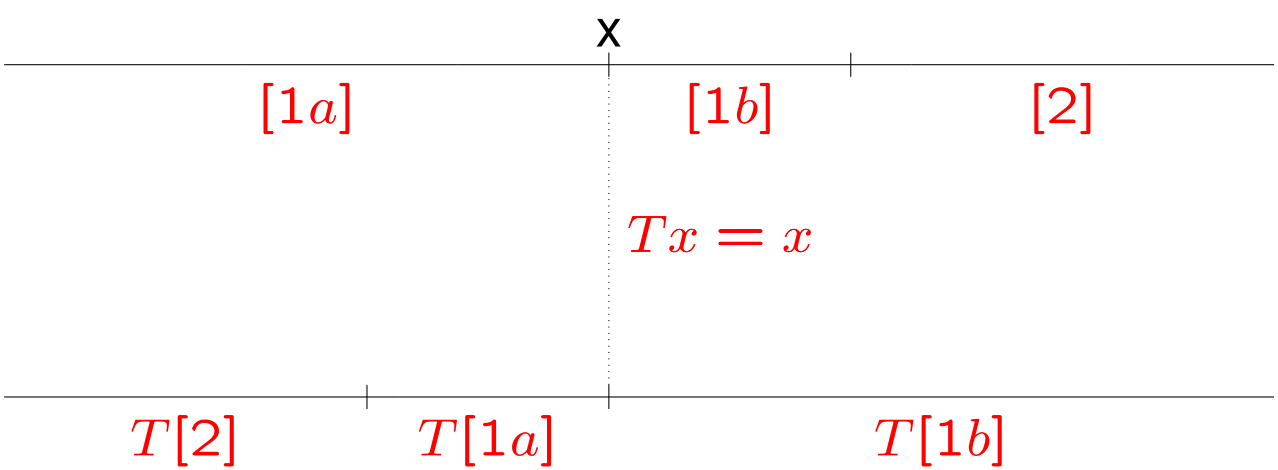


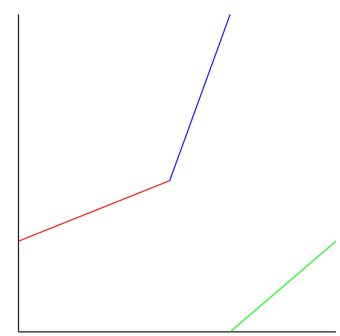


Is the order condition alone enough to make L the natural coding of an affine iet?
NO.

Counter-example Skew Sturmian: L generated by ...11112111....

BUT L is a grouped coding of an affine iet.





GENERALIZED IET

Is the order condition alone enough to make L a coding of an affine iet? NO.

Counter-example Skew Arnoux-Rauzy: L generated by the doubly infinite sequence $\bar{y}3y$,

y the Fibonacci sequence on 1 and 2, fixed point of the substitution $1 \to 12$, $2 \to 1$,

$\bar{y}$ its retrograde,

BUT L is the natural coding of a generalized iet (continuous increasing maps from $[j]$ to $T[j]$).

AND FINALLY

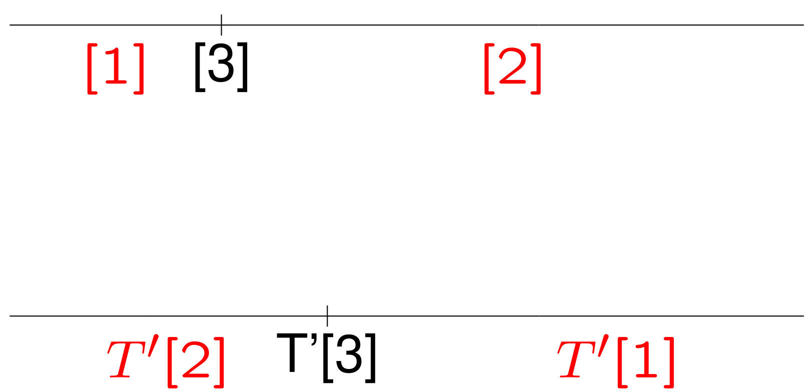
Main theorem L is the natural coding of a generalized iet <=> order condition.

Proof

We build canonically the language L' generated by (left and right) recurrent orbits, it is the natural coding of a standard iet T'.

We build T from T' by a Denjoy blow-up.

For SAR, T' is a rotation, and we blow up two points.



Note that for SS T' is the identity on I_1 coded by 1: the dream of the Marseille school.

AFFINE CRITERION

Theorem L is the natural coding of an affine iet =>

- order condition

- there exist θ_i, $i \in \mathcal{A}$, such that for each non recurrent bi-infinite word z in X_L,

$$\sum_{n \geq 0} \exp \sum_{j=0}^{n} \theta_{z_j} < +\infty. \tag{1}$$

$$\sum_{n > 0} \exp - \sum_{j=-n}^{-1} \theta_{z_j} < +\infty. \tag{2}$$

=> L is a grouped coding of an affine iet.

LOOKING FOR COUNTER-EXAMPLES

Affine natural coding but not standard. Possible from many aperiodic T', by Camelier-Guttierez, Cobo, Bressaud-Hubert-Maass, Marmi-Moussa-Yoccoz.

Grouped but not natural affine. Maybe only in (ultimately) periodic cases?

Generalized but not natural affine. When T' is the iet associated to some square-tiled surfaces, the Eierlegende Wollmilch Sau and the Ornithorynque.

TWO FAMILIES

Proposition L = natural coding of a generalized iet T,
L non recurrent,
L' aperiodic uniformly recurrent,
the two orders of L' are cyclically conjugate.
Then T cannot be of class P, hence is not affine.

Class P = except on a countable set of points, DT exists and $DT = h$ where h is a function with bounded variation, and $|h|$ is bounded from below by a strictly positive number.

Proposition L' = natural coding of a standard non purely periodic iet,
a, b letters,
$w_n = aw'_nb$ bispecial words in L',
u ending with w_n for all n,
v beginning with w_n for all n,
ω an extra symbol,
L generated by L' and $u\omega v$.
Then L is a natural coding of a generalized iet, but not a grouped (or natural) coding of an affine iet.

OPEN PROBLEMS

Conjecture Natural coding of an affine iet <=> order condition, conditions (1) and (2] for each non recurrent bi-infinite word.

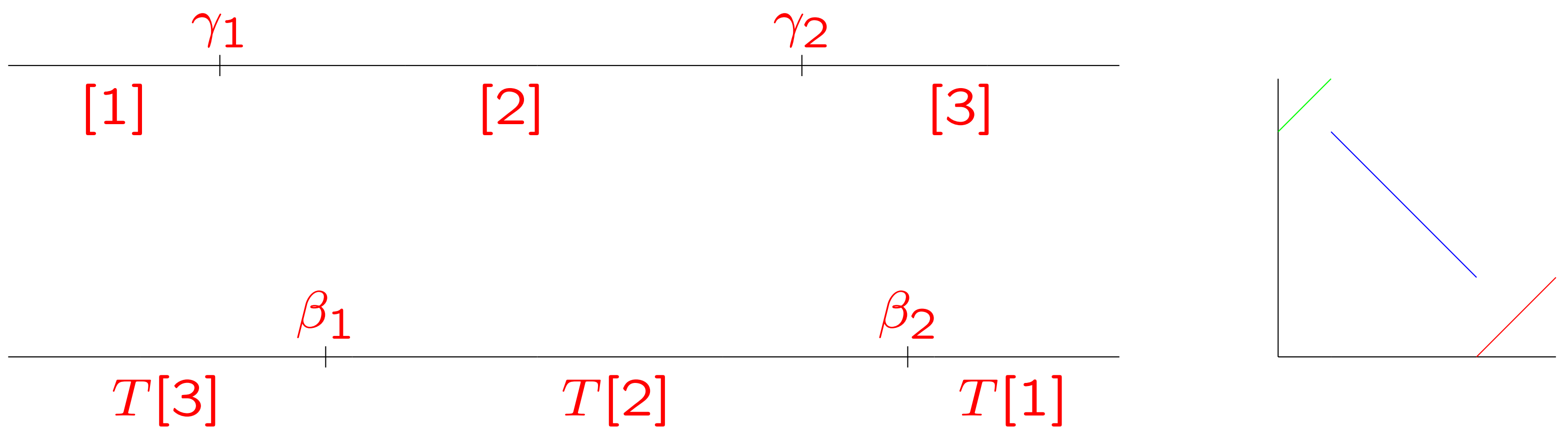
Example L generated by ...332332331331.... and ...413241324132...

Question Grouped coding of an affine iet <=> order condition,
condition (1) for each non recurrent bi-infinite word which is not ultimately periodic to the right,
condition (2) for each non recurrent bi-infinite word which is not ultimately periodic to the left ?

Conjecture Every word in a iet language is a word of a natural coding of an affine iet.

Question Languages of piecewise isometries?

IET WITH FLIPS



All results will remain, with a flipped order condition: for a bispecial w, write the order condition with $<_A$ and

- $<_D$ if w contains an even number of flipped letters
- $<_D$ reversed otherwise.