A different family of graphs to characterize dendric shift spaces

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Two families of graphs

Extensions

Left and right extensions:

$$E_X^L(w) = \{a \in \mathcal{A} \mid aw \in \mathcal{L}(X)\}, \quad E_X^R(w) = \{b \in \mathcal{A} \mid wb \in \mathcal{L}(X)\}$$

If $\#E_X^L(w) \geq 2$, w is said to be *left special*.

If $\#E_X^R(w) \geq 2$, w is said to be *right special*.

If w is left and right special, w is said to be *bispecial*.

Bi-extensions:

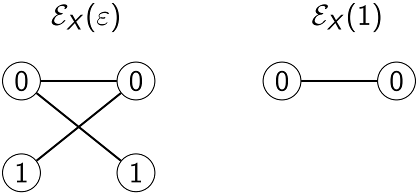
$$E_X(w) = \{(a, b) \in E_X^L(w) \times E_X^R(w) \mid awb \in \mathcal{L}(X)\}$$

Extension graphs

Definition

The *extension graph* of $w \in \mathcal{L}(X)$ is the bipartite graph $\mathcal{E}_X(w)$ with vertices $E^L_X(w) \sqcup E^R_X(w)$ and edges $E_X(w)$.

If X is the Fibonacci shift space,



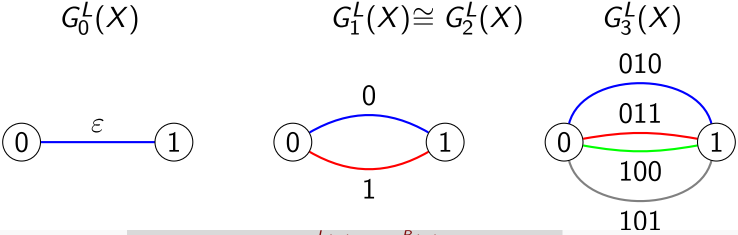
Graphs $G_n^L(X)$ and $G_n^R(X)$

Definition

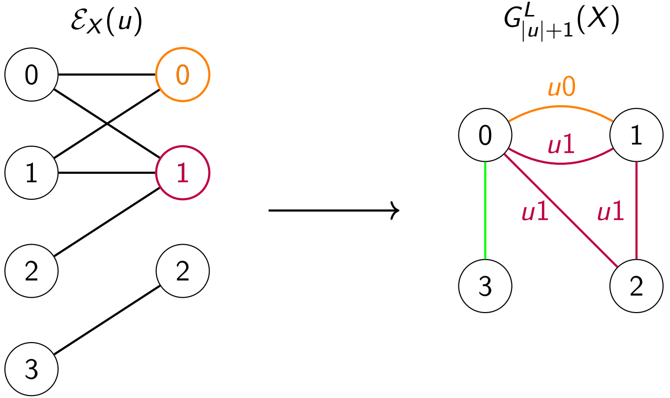
For $n \in \mathbb{N}$, $G_n^{L(resp.R)}(X)$ is defined as follows:

- the vertices are the elements of $\mathcal{A}$,
- there is an edge labeled by $v \in \mathcal{L}_n(X)$ between a and b if $a, b \in E^{L(resp.R)}(v)$.

If X is the Tribonacci shift space, $G_n^L(X)$ is a clique of size 3.
If X is the Thue-Morse shift space,



Link between the two



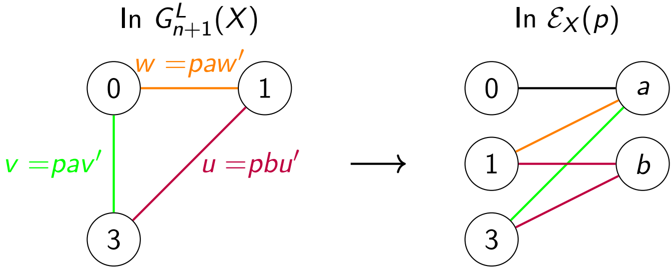
Lemma

$\mathcal{E}_X(u)$ contains the path $a_1^L \to b_1^R \to a_2^L \cdots \to b_k^R \to a_{k+1}^L$ iff $G^L_{|u|+1}(X)$ contains the path $a_1 \xrightarrow{ub_1} a_2 \ldots \xrightarrow{ub_k} a_{k+1}$.

Acyclicity

Observations:

- If $\mathcal{E}_X(u)$ contains a cycle, then $G^L_{|u|+1}(X)$ contains a cycle using edges of different colors.
- If $G^L_{n+1}(X)$ contains a cycle using edges of different colors, then there exists $p \in \mathcal{L}_{\leq n}(X)$ such that $\mathcal{E}_X(p)$ contains a cycle.



Acyclicity (2)

Definition

A graph with colored edges is *acyclic for the coloring* if any cycle only uses edges of one color.

Proposition

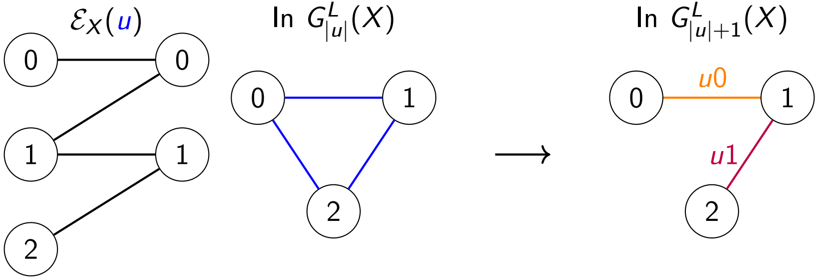
Let $N \in \mathbb{N}$. The following are equivalent:

1. *for all $u \in \mathcal{L}_{\leq N}(X)$, $\mathcal{E}_X(u)$ is acyclic,*
2. *for all $n \leq N+1$, $G_n^L(X)$ is acyclic for the coloring,*
3. *for all $n \leq N+1$, $G_n^R(X)$ is acyclic for the coloring.*

Consequences of acyclicity

If $\mathcal{E}_X(u)$ is acyclic for all $u \in \mathcal{L}_{\leq N}(X)$, then

- $G^L_{n+1}(X)$ is a "subgraph" of $G^L_n(X)$,
- if $\mathcal{E}_X(u)$ contains the path $a_1^L \to b_1^R \to a_2^L \cdots \to b_{k-1}^R \to a_k^L$, then for all $n \geq |u| + 1$, any path connecting a_1 to a_k in $G^L_n(X)$ passes through $a_2, \ldots, a_{k-1}$.



Connectedness

Proposition

Let $N \in \mathbb{N}$. If $\mathcal{E}_X(u)$ is acyclic for all $u \in \mathcal{L}_{\leq N}(X)$, then the following are equivalent:

1. *for all $u \in \mathcal{L}_{\leq N}(X)$, $\mathcal{E}_X(u)$ is connected,*
2. *for all $n \leq N+1$, $G^L_n(X)$ is connected,*
3. *$G^L_{N+1}(X)$ is connected,*
4. *for all $n \leq N+1$, $G^R_n(X)$ is connected,*
5. *$G^R_{N+1}(X)$ is connected.*

Dendricity

Definition (Berthé, De Felice, Dolce, Leroy, Perrin, Reutenauer, Rindone)

A word $u \in \mathcal{L}(X)$ is *dendric* (in X) if $\mathcal{E}_X(u)$ is a tree.

A shift space X is *dendric* if all the elements of $\mathcal{L}(X)$ are dendric.

Proposition

The following are equivalent:

1. *X is dendric,*
2. *for all $n \in \mathbb{N}$, $G_n^L(X)$ is acyclic for the coloring and connected,*
3. *for all $n \in \mathbb{N}$, $G_n^R(X)$ is acyclic for the coloring and connected.*

Limit behavior

Stabilization

Proposition (Consequence of Dolce, Perrin)

If X is eventually dendric, there exists N such that for all $n \geq N$,

$$G_n^L(X) \cong G_N^L(X) \quad \textit{and} \quad G_n^R(X) \cong G_N^R(X).$$

A shift space X is *eventually dendric* if all the long enough elements of $\mathcal{L}(X)$ are dendric.

Definition

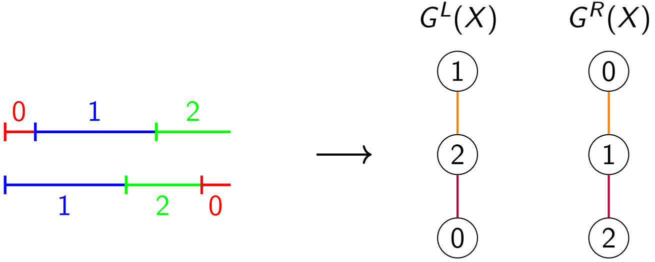
If it exists, we denote

$$G^L(X) \cong \lim_n G_n^L(X) \quad \text{and} \quad G^R(X) \cong \lim_n G_n^R(X).$$

Examples

Arnoux-Rauzy:
The graphs $G^L(X)$ and $G^R(X)$ are cliques of size $\#\mathcal{A}$.

Interval exchanges:
The graphs $G^L(X)$ and $G^R(X)$ are line graphs.



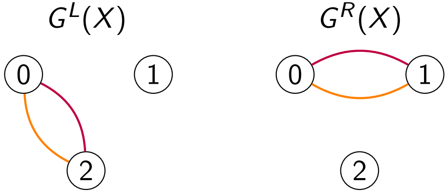
Chacon ternary shift

The Chacon ternary shift space X is generated by the morphism

$$\sigma : 0 \mapsto 0012, 1 \mapsto 12, 2 \mapsto 012.$$

- X is not eventually dendric. [Dolce, Perrin]
- The factor complexity of X is $p_n(X) = 2n + 1$. [Ferenczi]

$E_X^L(0) = \{0, 2\} = E_X^L(1)$ and $E_X^R(0) = \{0, 1\} = E_X^L(2)$ thus



Asymptotic pairs

In X is eventually dendric, the $G^L(X)$ is also defined as follows:

- the vertices are the elements of $\mathcal{A}$,
- there is an edge labeled by $x \in \mathcal{A}^{\mathbb{N}}$ between a and b if there exist $y, y' \in \mathcal{A}^{-\mathbb{N}}$ such that $yax, y'bx \in X$.

For the Chacon shift space, the two definitions are different.

S-adic characterization of dendric shift spaces

Return morphisms

Definition

A *return morphism* for $w \neq \varepsilon$ is an injective morphism $\sigma : \mathcal{A}^* \to \mathcal{B}^*$ such that, for all $a \in \mathcal{A}$,

$$|\sigma(a)w|_w = 2, \quad \sigma(a)w \in w\mathcal{B}^*.$$

$$\sigma : \begin{cases} 0 \mapsto 010 \\ 1 \mapsto 0210 \\ 2 \mapsto 0222210 \end{cases} \qquad \tau : \begin{cases} 0 \mapsto 0101 \\ 1 \mapsto 01001 \\ 2 \mapsto 01021001 \end{cases}$$

S-adic representations

Definition

A primitive *S-adic representation* of a minimal shift space X is a primitive sequence of morphisms $(\sigma_n : \mathcal{A}_{n+1}^* \to \mathcal{A}_n^*)_n$ such that

$$\mathcal{L}(X) = \bigcup_N \mathrm{Fac}(\sigma_0 \dots \sigma_N(\mathcal{A}_{N+1})).$$

Proposition (Berthé *et al.*)

Every minimal dendric shift space over $\mathcal{A}$ has a primitive S-adic representation such that

- *the morphisms are return morphisms over $\mathcal{A}$,*
- *the intermediary shift spaces are dendric.*

Dendric images

Theorem (G., Lejeune, Leroy)

Let X be a dendric shift space and σ a return morphism for w. Then $\sigma(X)$ is dendric if and only if

- 'I' *for all $u \in \mathcal{L}(\sigma(\mathcal{A})w)$ st. $|u|_w = 0$, u is dendric in $\mathcal{L}(\sigma(\mathcal{A})w)$,*
- 'L' *for all $s \in \mathcal{A}^*$ and for all $v \in \mathcal{L}(X)$, when removing in $\mathcal{E}_X(v)$ the left vertices in*

  $$\mathcal{A}_s^L := \{a \in \mathcal{A} \mid \sigma(a) \notin \mathcal{A}^* s\},$$

  the other left vertices remain connected,
- 'R' *for all $p \in \mathcal{A}^*$ and for all $v \in \mathcal{L}(X)$, when removing in $\mathcal{E}_X(v)$ the right vertices in*

  $$\mathcal{A}_p^R := \{a \in \mathcal{A} \mid \sigma(a)w \notin p\mathcal{A}^*\},$$

  the other right vertices remain connected.

Dendric images using $G^L(X)$ and $G^R(X)$

Theorem

Let X be a dendric shift space and σ a return morphism for w. Then $\sigma(X)$ is dendric if and only if

- ‘I’ *for all $u \in \mathcal{L}(\sigma(\mathcal{A})w)$ st. $|u|_w = 0$, u is dendric in $\mathcal{L}(\sigma(\mathcal{A})w)$,*
- ‘L’ *for all $s \in \mathcal{A}^*$, when removing in $G^L(X)$ the vertices in $\mathcal{A}^L_s$, the graph remains connected,*
- ‘R’ *for all $p \in \mathcal{A}^*$, when removing in $G^R(X)$ the vertices in $\mathcal{A}^R_p$ the graph remains connected.*

Image graph

Proposition

Let σ be a return morphism and let G be a colored graph.
We can define in a constructive way the graphs

- $\sigma^L \cdot G$ *such that, for all eventually dendric shift space* X *with* $G^L(X) = G$, *we have* $G^L(\sigma(X)) = \sigma^L \cdot G$,
- $\sigma^R \cdot G$ *such that, for all eventually dendric shift space* X *with* $G^R(X) = G$, *we have* $G^R(\sigma(X)) = \sigma^R \cdot G$.

S-adic characterization

Let $\mathcal{S}$ be a set of return morphisms over $\mathcal{A}$ satisfying 'I'. The graph $\mathcal{G}^{L(resp.R)}(\mathcal{S})$ is defined as follows:

- its set of vertices is $\{G^{L(resp.R)}(X) \mid X \text{ dendric shift space over } \mathcal{A}\}$,
- there is an edge labeled by $\sigma \in \mathcal{S}$ from $\sigma^{L(resp.R)} \cdot G$ to G if G satisfies condition 'L'(resp. 'R') for σ.

Theorem

Let X be a shift space having a primitive S-adic representation $\boldsymbol{\sigma} = (\sigma_n)_n$ where σ_n is a return morphism for all n. If $\mathcal{S} = \{\sigma_n : n \in \mathbb{N}\}$, then X is dendric if and only if $\boldsymbol{\sigma}$ labels infinite paths in both $\mathcal{G}^L(\mathcal{S})$ and $\mathcal{G}^R(\mathcal{S})$.

End

Thank you for your attention!