Decidable properties of substitution shifts

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Overview

We investigate the decidability of many properties of substitution shifts like:

- periodicity,
- recognizability,
- minimality.

We work under the most general hypotheses with morphisms which are not supposed primitive and can be even erasing.

Shift spaces

Let A be a finite alphabet. The shift transformation $S : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is defined by $y = S(x)$ if $y_n = x_{n+1}$ for every $n \in \mathbb{Z}$.
A shift space is a closed and shift-invariant subset of $A^{\mathbb{Z}}$. We denote by $\mathcal{L}(X)$ the set of factors of points in a shift space X, called the language of X. We also denote $\mathcal{L}(x)$ the set of factors of $x \in A^{\mathbb{Z}}$.
A nonempty shift space X is minimal if it does not contain strictly another nonempty shift space.
A point $x \in A^{\mathbb{Z}}$ is periodic if $S^n(x) = x$ for some $n \geq 1$. A shift space is aperiodic if it does not contain a periodic point.

Substitution shifts

A morphism $\sigma \colon A^* \to A^*$ is a monoid morphism from A^* to A^*.
It is non-erasing if $\sigma(a)$ is nonempty for every $a \in A$.
A word w is growing if $\lim |\sigma^n(v)| = \infty$.
The language of σ, denoted $\mathcal{L}(\sigma)$ is the set of factors of the words $\sigma^n(a)$ for $n \geq 0$.

The substitution shift defined by σ, denoted by $X(\sigma)$ is the shift space $X = \{x \in A^{\mathbb{Z}} \mid \mathcal{L}(x) \subset \mathcal{L}(\sigma)\}$.
A morphism $\sigma \colon A^* \to A^*$ is primitive if there is an integer $n \geq 1$ such that every $b \in A$ appears in every $\sigma^n(a)$ for $a \in A$. It is aperiodic if $X(\sigma)$ is aperiodic.
A primitive shift space is minimal.

The language $\mathcal{L}(X(\sigma))$

One has $\mathcal{L}(X(\sigma)) \subset \mathcal{L}(\sigma)$ but the converse is false in general.

Example

If $\sigma\colon a \mapsto ab, b \mapsto b$, then $\mathcal{L}(\sigma) = ab^* \cup b^*$ but $X(\sigma) = b^\infty$ and thus $a \notin \mathcal{L}(X(\sigma))$.

The following is related to a result of Salo (2017) who proved that $\mathcal{L}(\sigma)$ is decidable.

Theorem

Let $\sigma\colon A^* \to A^*$ be a morphism. The language $\mathcal{L}(X(\sigma))$ is decidable.

Fixed points

A sequence x is a fixed point of a morphism σ if $\sigma(x) = x$. It is admissible if $x \in X(\sigma)$.

Theorem

A morphism $\sigma \colon A^* \to A^*$ such that $X(\sigma)$ is nonempty has a power σ^k with an admissible two-sided infinite fixed point. The set of orbits of these fixed points is a finite computable set and the exponent k can be bounded in terms of σ.

A morphism $\sigma \colon A^* \to A^*$ is right prolongable on $u \in A^+$ if $\sigma(u)$ begins with u and u is growing. In this case, there is a unique right-infinite word x which has each $\sigma^n(u)$ as a prefix. We denote $x = \sigma^\omega(u)$. Symmetrically, σ is left prolongable on $v \in A^+$ if $\sigma(v)$ ends with v and v is growing. We denote by $\sigma^{\tilde{\omega}}(v)$ the left infinite sequence which has all $\sigma^n(v)$ as suffixes.

The number of fixed points may be infinite as shown in the example below.

Example

Let $\sigma\colon a \mapsto bc, b \mapsto bd, c \mapsto ec, d \mapsto d, e \mapsto e$. Then, for any $n \geq 0$, the sequences ${}^{\omega}de^{n} \cdot e^{\omega}$ and ${}^{\omega}d \cdot d^{n}e^{\omega}$ are admissible two-sided fixed points of σ.

We denote $F(\sigma) = \{w \in A^* \mid \sigma(w) = w\}$.

Proposition

Let $\sigma : A^* \to A^*$ be a morphism. A sequence $x \in X(\sigma)$ is a two-sided infinite fixed point of σ if and only if one of the following conditions is satisfied.

(i) $x = {}^{\omega}rs \cdot tu^{\omega}$ *with* $r, s, t, u \in F(\sigma)^*$ *and* $r^*stu^* \subset \mathcal{L}(\sigma)$.

(ii) $x = {}^{\omega}rs \cdot \sigma^{\omega}(t)$ *with* $r, s \in F(\sigma)^*$, σ *right prolongable on* t *and* $r^*st \subset \mathcal{L}(\sigma)$.

(iii) $x = \sigma^{\tilde{\omega}}(r) \cdot st^{\omega}$ *with* σ *left prolongable on* r, $s, t \in F(\sigma)^*$ *and* $rst^* \subset \mathcal{L}(\sigma)$.

(iv) $x = \sigma^{\tilde{\omega}}(r) \cdot \sigma^{\omega}(s)$ *with* σ *left prolongable on* r *and right prolongable on* s *and* $rst \in \mathcal{L}(\sigma)$.

Quasi-fixed points

A two-sided infinite sequence x is a quasi-fixed point if $\sigma(x) = S^k x$ for some $k \in \mathbb{Z}$. A quasi-fixed point x is admissible if x is in $X(\sigma)$.
If a is a letter such that $\sigma(a) = uav$ with u, v non-erasable and $i = |ua|$, we denote

$$\sigma^{\omega,i}(a) = \cdots \sigma^2(u)\sigma(u)u \cdot av\sigma(v)\sigma^2(v)\cdots$$

Example

Let $\sigma\colon a \mapsto bab, b \mapsto b$. Then $\sigma^{\omega,2}(a) = {}^{\omega} b \cdot ab^{\omega}$ is a quasi-fixed point of σ.

Theorem

Let $\sigma\colon A^* \to A^*$ *be a morphism. A two-sided infinite sequence* $x \in X(\sigma)$ *is a quasi-fixed point of* σ *if and only if one of the following conditions is satisfied.*

(i') x *is a shift of an admissible fixed point of* σ.

(ii') $x \sim \sigma^{\omega,i}(a)$ *or some* $a \in A$.

(iii') $x = (uv)^\infty$ *for some nonempty words* u, v *such that* $\sigma(uv) = vu$.

There is a finite and computable number of orbits of admissible quasi-fixed points of powers of σ.

Periodicity

A sequence x is periodic if $S^n x = x$ for some $n \geq 1$. A shift space is periodic if is formed of periodic points. It is aperiodic if it contains no periodic point.

The following was proved by Pansiot (1986) and also by Harju and Linna (1986) for a primitive morphism

Theorem

It is decidable whether a morphism $\sigma \colon A^* \to A^*$ is aperiodic. The set of periodic points in $X(\sigma)$ is finite and their periods are effectively bounded in terms of σ.

Theorem

It is decidable whether the shift $X(\sigma)$ generated by a morphism σ is periodic.

The proof uses the fact that the shift $X(\sigma)$ is periodic if and only if every admissible quasi-fixed point of a power of σ is periodic.

Elementary morphisms

A morphism $\sigma\colon A^* \to C^*$ is *elementary* if for every decomposition $\sigma = \alpha \circ \beta$ with $\beta\colon A^* \to B^*$ and $\alpha\colon B^* \to C^*$, one has $\mathrm{Card}(B) \geq \mathrm{Card}(A)$. An elementary morphism is clearly non-erasing.
The following result is due to Pansiot (1986).

Lemma

Let $\sigma\colon A^* \to A^*$ be an elementary morphism having a periodic point $x \in X(\sigma)$. The minimal period of x can be effectively bounded in terms of the morphism σ.

When σ is growing, the period of x is bounded by $\mathrm{Card}(A)$.

Example

The morphism $\sigma\colon a \mapsto a, b \mapsto baab$, is elementary. Since $\sigma(aab) = (aab)^2$, the sequence $x = (aab)^\infty$ is a fixed point of σ.

Recognizability

Let $\sigma\colon A^* \to B^*$ be a morphism. A *σ-representation* of $y \in B^{\mathbb{Z}}$ is a pair (x, k) of a sequence $x \in A^{\mathbb{Z}}$ and an integer k such that

$$y = S^k(\sigma(x)). \tag{1}$$

The σ-representation (x, k) is *centered* if $0 \leq k < |\sigma(x_0)|$.
Let $\sigma\colon A^* \to B^*$ be a morphism and let X be a shift space on A. The morphism σ is recognizable on X at a point $y \in B^{\mathbb{Z}}$ if y has at most one centered σ-representation (x, k) with $x \in X$. It is recognizable on X if it is recognizable on X for every point $y \in B^{\mathbb{Z}}$.

Theorem

Every morphism σ is recognizable on $X(\sigma)$ for aperiodic points.

This result has a substantial history.

It was first proved by Mossé (1992, 1996) that every aperiodic primitive morphism is recognizable on $X(\sigma)$.

Later, this important result was generalized by Bezuglyi, Kwiatkowski, Medynets (2009) who showed that every aperiodic non-erasing morphism σ is recognizable on $X(\sigma)$. As a further extension, Berthe, Steiner, Thuswaldner, Yassawi (2019) showed that a non-erasing morphism σ is recognizable on $X(\sigma)$ at aperiodic points.

Finally, the last result was extended by Beal, P. and Restivo (2021) to arbitrary morphisms.

We prove, using the decidability of periodicity the following decidability property.

Theorem

It is decidable whether a morphism σ is recognizable on $X(\sigma)$.

Minimality

The following statement was proved by Maloney, Rust (2018) with the additional hypothesis $\mathcal{L}(X(\sigma)) = \mathcal{L}(\sigma)$.

Theorem

A morphism σ is minimal if and only if one of the following conditions is satisfied.

(i) $X(\sigma)$ *is the closure under the shift of* w^∞ *with* $\sigma(w) = w$.

(ii) *there is a growing letter* $a \in \mathcal{L}(X(\sigma))$ *which appears with bounded gaps in* $\mathcal{L}(\sigma)$ *and for every* $b \in \mathcal{L}(X(\sigma))$ *there is some* $n \geq 1$ *such that* $|\sigma^n(a)|_b \geq 1$.

The previous theorem allows us to prove the following result. It is actually related with the result of Durand (2013b) which states that is is decidable whether the image by a morphism ϕ of an admissible one-sided fixed point of a morphism σ is uniformly recurrent.

Theorem

It is decidable whether a morphism is minimal.

Open problems

1. Is it decidable whether a substitution shift $X(\sigma)$ is also of the form $X(\tau)$ with τ of constant length?
2. A shift is morphic if it is the closure under the shift of $\phi(X(\sigma))$ with ϕ letter-to letter. Is it decidable whether a morphic shift is purely morphic, that is, a substitution shift?
3. Is it decidable whether a substitution shift is of the form $X(\sigma, \phi)$ with σ, ϕ non-erasing?
4. Is the inclusion problem $X(\sigma) \subset X(\tau)$ decidable? It is decidable for minimal shifts (Durand, Leroy, 2022).
5. What is the situation in $2D$?