Sous-shifts avec trous

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Subshift with a (lexicographic) hole

$$\Omega_{\mathbf{a},\mathbf{b}} := \left\{\mathbf{u} \in \{0,1\}^\infty \ : \ \Sigma^n(\mathbf{u}) \notin (\mathbf{a},\mathbf{b}) \ \ \forall n \geq 0\right\}$$
$$\mathbf{a} \in 0\{0,1\}^\infty, \ \mathbf{b} \in 1\{0,1\}^\infty, \Sigma \text{ shift map},$$
$$(\mathbf{a},\mathbf{b}) = \{\mathbf{u} \in \{0,1\}^\infty : \mathbf{a} <_{\text{lex}} \mathbf{u} <_{\text{lex}} \mathbf{b}\}$$

$$\Omega_{\mathbf{a},\mathbf{b}} = \left\{i_1 i_2 \cdots \in \{0,1\}^\infty \ : \ i_n i_{n+1} \cdots \leq_{\text{lex}} \mathbf{a} \text{ if } i_n = 0, \right.$$
$$\left. i_n i_{n+1} \cdots \geq_{\text{lex}} \mathbf{b} \text{ if } i_n = 1\right\}$$

Examples:

$\Omega_{\overline{0},\overline{1}} = \{\overline{0},\overline{1}\}$, $\overline{w} := w^\infty = www\cdots$

$\Omega_{0\overline{1},1\overline{0}} = \{0,1\}^\infty$

$\Omega_{\overline{01},\overline{10}} = 0^*\overline{01} \cup 1^*\overline{10} \cup \{\overline{0},\overline{1}\}$

$$\Theta_{\mathbf{a},\mathbf{b}} := \left\{\mathbf{u} \in \{0,1\}^\infty \ : \ \Sigma^n(\mathbf{u}) \in [\mathbf{a},\mathbf{b}] \ \ \forall n \geq 0\right\}$$
$$\Omega_{0\mathbf{b},1\mathbf{a}} = 0^*\Theta_{\mathbf{a},\mathbf{b}} \cup 1^*\Theta_{\mathbf{a},\mathbf{b}} \cup \{\overline{0},\overline{1}\}$$

Thue–Morse–Sturmian substitutions, S-adic words

$$L: \begin{array}{l} 0 \mapsto 0 \\ 1 \mapsto 10 \end{array} \qquad M: \begin{array}{l} 0 \mapsto 01 \\ 1 \mapsto 10 \end{array} \qquad R: \begin{array}{l} 0 \mapsto 01 \\ 1 \mapsto 1 \end{array}$$

limit word (or S-adic word)

$$\boldsymbol{\sigma} = (\sigma_n)_{n\geq 1} \in \{L, M, R\}^\infty,\ \mathbf{u} \in \{0,1\}^\infty$$
$$\boldsymbol{\sigma}(\mathbf{u}) := \lim_{n\to\infty} \sigma_1\sigma_2\cdots\sigma_n(\mathbf{u})$$

exists because $L(i), M(i), R(i)$ start with i for $i \in \{0,1\}$

$\boldsymbol{\sigma} \in \{L, M, R\}^\infty$ primitive if and only if $\boldsymbol{\sigma}$ does not end with $\overline{L}$ or $\overline{R}$

Theorem (Labarca–Moreira '06, Glendinning–Sidorov '15, St–Komornik–Zou)

Let $\mathbf{a} \in 0\{0,1\}^\infty$, $\mathbf{b} \in 1\{0,1\}^\infty$.

(i) $\Omega_{\mathbf{a},\mathbf{b}} \neq \{\bar{0}, \bar{1}\}$ *if and only if* $\mathbf{a} = 0\bar{1}$, *or* $\mathbf{b} = 1\bar{0}$, *or*
$\mathbf{a} \geq \sigma(\bar{0})$, $\mathbf{b} \leq \sigma(\bar{1})$ *for some* $\sigma \in \{L,R\}^* M$, *or*
$\mathbf{a} = \boldsymbol{\sigma}(\bar{0})$, $\mathbf{b} = \boldsymbol{\sigma}(\bar{1})$ *for some primitive* $\boldsymbol{\sigma} \in \{L,R\}^\infty$.

(ii) $\Omega_{\mathbf{a},\mathbf{b}} = \{\bar{0}, \bar{1}\}$ *if and only if*
$\mathbf{a} < \sigma(\bar{0})$, $\mathbf{b} \geq \sigma(1\bar{0})$ *for some* $\sigma \in \{L,R\}^* M$, *or*
$\mathbf{a} \leq \sigma(0\bar{1})$, $\mathbf{b} > \sigma(\bar{1})$ *for some* $\sigma \in \{L,R\}^* M$.

(iii) $\Omega_{\mathbf{a},\mathbf{b}}$ *is uncountable with positive entropy if and only if*
$\mathbf{a} \geq \sigma(\bar{0})$, $\mathbf{b} < \sigma(1\bar{0})$ *for some* $\sigma \in \{L,M,R\}^* M$, *or*
$\mathbf{a} > \sigma(0\bar{1})$, $\mathbf{b} \leq \sigma(\bar{1})$ *for some* $\sigma \in \{L,M,R\}^* M$.

(iv) $\Omega_{\mathbf{a},\mathbf{b}}$ *is uncountable with zero entropy if and only if*
$\mathbf{a} = \boldsymbol{\sigma}(\bar{0})$, $\mathbf{b} = \boldsymbol{\sigma}(\bar{1})$ *for some primitive* $\boldsymbol{\sigma} \in \{L,M,R\}^\infty$.

(v) $\Omega_{\mathbf{a},\mathbf{b}}$ *is countable if and only if*
$\mathbf{a} \leq \sigma(01\bar{0})$, $\mathbf{b} \geq \sigma(1\bar{0})$ *for some* $\sigma \in \{L,M,R\}^*$, *or*
$\mathbf{a} \leq \sigma(0\bar{1})$, $\mathbf{b} \geq \sigma(10\bar{1})$ *for some* $\sigma \in \{L,M,R\}^*$.

Elements of the proof

- σ order-preserving for all $\sigma \in \{L, M, R\}^*$

$$L : 0 \mapsto 0 \quad M : 0 \mapsto 01 \quad R : 0 \mapsto 01$$
$$1 \mapsto 10 \quad 1 \mapsto 10 \quad 1 \mapsto 1$$

- $\mathbf{u} \in \Omega_{\sigma(\mathbf{a}),\sigma(\mathbf{b})} \setminus \{\bar{0}, \bar{1}\},\ \sigma \in \{L, M, R\} \implies \mathbf{u} \in \{0,1\}^* \sigma(\Omega_{\mathbf{a},\mathbf{b}})$

$$\Omega_{\mathbf{a},\mathbf{b}} = \left\{\mathbf{u} \in \{0,1\}^\infty : \Sigma^n(\mathbf{u}) \notin (\mathbf{a}, \mathbf{b}) \ \ \forall n \geq 0\right\}$$

$$\mathbf{u} \in \Omega_{L(0\bar{1}),L(1\bar{0})} = \Omega_{\overline{01},1\bar{0}} \ \Rightarrow \ \mathbf{u} \in 1^*\{0, 10\}^\infty \cup \{\bar{1}\}$$
$$\mathbf{u} \in \Omega_{R(0\bar{1}),R(1\bar{0})} = \Omega_{0\bar{1},\overline{10}} \ \Rightarrow \ \mathbf{u} \in 0^*\{1, 01\}^\infty \cup \{\bar{0}\}$$
$$\mathbf{u} \in \Omega_{M(0\bar{1}),M(1\bar{0})} = \Omega_{01\overline{10},10\overline{01}} \ \Rightarrow \ \mathbf{u} \in 0^*\{01, 10\}^\infty \cup 1^*\{01, 10\}^\infty \cup \{\bar{0}, \bar{1}\}$$

- $\Omega_{M(\bar{0}),M(\bar{1})} = 0^*\overline{01} \cup 1^*\overline{10} \cup \{\bar{0}, \bar{1}\} \neq \{\bar{0}, \bar{1}\}$

- $\Omega_{M(\bar{0}),\mathbf{b}}$, $\mathbf{b} < M(1\bar{0})$, contains $\{0(01)^k, 0(01)^{k+1}\}^\infty$ for some $k \geq 0$,
  $\Omega_{\mathbf{a},M(\bar{1})}$, $\mathbf{a} > M(0\bar{1})$, contains $\{1(10)^k, 1(10)^{k+1}\}^\infty$ for some $k \geq 0$
  $\Rightarrow$ positive entropy

Partitions

$$\big(\sigma(01\bar{0}),\sigma(0\bar{1})\big) = \underbrace{\big(\sigma(01\bar{0}),\sigma(\overline{01})\big)}_{(\sigma L(01\bar{0}),\sigma L(0\bar{1}))} \cup \underbrace{\big[\sigma(\overline{01}),\sigma(01\overline{10})\big]}_{[\sigma M(\bar{0}),\sigma M(0\bar{1})]} \cup \underbrace{\big(\sigma(01\overline{10}),\sigma(0\bar{1})\big)}_{(\sigma R(01\bar{0}),\sigma R(0\bar{1}))}$$

$$(01\bar{0},0\bar{1}) = \bigcup_{\sigma\in\{L,R\}^* M} [\sigma(\bar{0}),\sigma(0\bar{1})] \cup \bigcup_{\boldsymbol{\sigma}\in\{L,R\}^\infty \text{ primitive}} \{\boldsymbol{\sigma}(\bar{0})\}$$

$$\big(\sigma(01\bar{0}),\sigma(0\bar{1})\big) = \underbrace{\big(\sigma(01\bar{0}),\sigma(\overline{01})\big)}_{(\sigma L(01\bar{0}),\sigma L(0\bar{1}))} \cup \underbrace{\big[\sigma(\overline{01}),\sigma(0110\overline{01})\big]}_{[\sigma M(\bar{0}),\sigma M(01\bar{0})]}$$

$$\cup \underbrace{\big(\sigma(0110\overline{01}),\sigma(01\overline{10})\big)}_{(\sigma M(01\bar{0}),\sigma M(0\bar{1}))} \cup \Big\{\underbrace{\sigma(01\overline{10})}_{\sigma M(0\bar{1})}\Big\} \cup \underbrace{\big(\sigma(01\overline{10}),\sigma(0\bar{1})\big)}_{(\sigma R(01\bar{0}),\sigma R(0\bar{1}))}$$

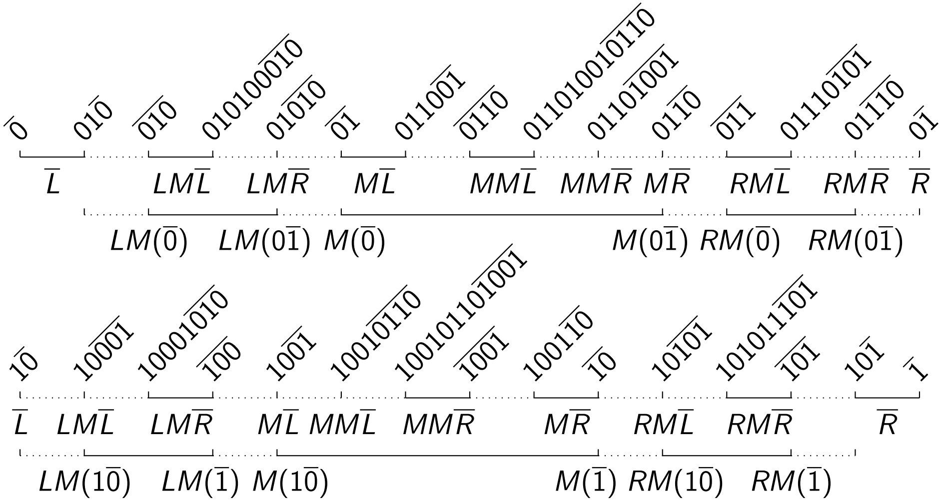
$$(01\bar{0},0\bar{1}) = \bigcup_{\sigma\in\{L,M,R\}^* M} \big([\sigma(\bar{0}),\sigma(01\bar{0})] \cup \{\sigma(0\bar{1})\}\big) \cup \bigcup_{\boldsymbol{\sigma}\in\{L,M,R\}^\infty \text{ primitive}} \{\boldsymbol{\sigma}(\bar{0})\}$$

$$[\bar{0},0\bar{1}] = \bigcup_{\boldsymbol{\sigma}\in\{L,M,R\}^\infty\setminus\{L,M,R\}^*\{L\bar{R},R\bar{L}\}} [\boldsymbol{\sigma}(\bar{0}),\boldsymbol{\sigma}(0\bar{1})]$$

$$\boldsymbol{\sigma} <_{\text{lex}} \boldsymbol{\tau} \Rightarrow \boldsymbol{\sigma}(0\bar{1}) <_{\text{lex}} \boldsymbol{\tau}(\bar{0}) \quad (\boldsymbol{\sigma},\boldsymbol{\tau} \in \{L,M,R\}^\infty \setminus \{L,M,R\}^*\{L\bar{R},R\bar{L}\})$$

$$s:\ \{0,1\}^\infty \to \{L,M,R\}^\infty \setminus \{L,M,R\}^*\{L\overline{R}, R\overline{L}\},$$
$$\mathbf{u} \mapsto \boldsymbol{\sigma} \quad \text{if } \mathbf{u} \in [\boldsymbol{\sigma}(\overline{0}), \boldsymbol{\sigma}(0\overline{1})] \cup [\boldsymbol{\sigma}(1\overline{0}), \boldsymbol{\sigma}(\overline{1})],$$

montonically increasing on $0\{0,1\}^\infty$, $1\{0,1\}^\infty$



Theorem (reformulation)

Let $\mathbf{a} \in 0\{0,1\}^\infty$, $\mathbf{b} \in 1\{0,1\}^\infty$.

(i) $\Omega_{\mathbf{a},\mathbf{b}} \neq \{\overline{0}, \overline{1}\}$ *if and only if* $\mathbf{a} = 0\overline{1}$, *or* $\mathbf{b} = 1\overline{0}$, *or*
$\mathbf{a} \geq \sigma(\overline{0})$, $\mathbf{b} \leq \sigma(\overline{1})$ *for some* $\sigma \in \{L, R\}^* M$, *or*
$\mathbf{a} = \boldsymbol{\sigma}(\overline{0})$, $\mathbf{b} = \boldsymbol{\sigma}(\overline{1})$ *for some primitive* $\boldsymbol{\sigma} \in \{L, R\}^\infty$.

(ii) $\Omega_{\mathbf{a},\mathbf{b}} = \{\overline{0}, \overline{1}\}$ *if and only if*
$\mathbf{a} < \sigma(\overline{0})$, $\mathbf{b} \geq \sigma(1\overline{0})$ *for some* $\sigma \in \{L, R\}^* M$, *or*
$\mathbf{a} \leq \sigma(0\overline{1})$, $\mathbf{b} > \sigma(\overline{1})$ *for some* $\sigma \in \{L, R\}^* M$.

(iii') $\Omega_{\mathbf{a},\mathbf{b}}$ *is uncountable with positive entropy if and only if* $s(\mathbf{a}) > s(\mathbf{b})$.

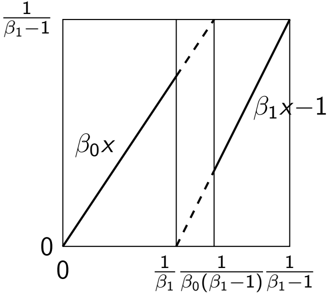
(iv') $\Omega_{\mathbf{a},\mathbf{b}}$ *is uncountable with zero entropy if and only if* $s(\mathbf{a}) = s(\mathbf{b})$ *is primitive.*

(v') $\Omega_{\mathbf{a},\mathbf{b}}$ *is countable if and only if* $s(\mathbf{a}) < s(\mathbf{b})$ *or* $s(\mathbf{a}) = s(\mathbf{b})$ *ends with* $\overline{L}$ *or* $\overline{R}$.

Unique double base expansions

$$\pi_{\beta_0,\beta_1}(i_1 i_2 \cdots) := \sum_{k=1}^{\infty} \frac{i_k}{\beta_{i_1}\beta_{i_2}\cdots\beta_{i_k}}$$

$$U_{\beta_0,\beta_1} := \left\{\mathbf{u} \in \{0,1\}^\infty \,:\, \pi_{\beta_0,\beta_1}(\mathbf{u}) \neq \pi_{\beta_0,\beta_1}(\mathbf{v}) \text{ for all } \mathbf{v} \neq \mathbf{u}\right\}$$



$$U_{\beta_0,\beta_1} = \left\{\mathbf{u} \in \{0,1\}^\infty \,:\, \pi_{\beta_0,\beta_1}(\Sigma^n \mathbf{u}) \notin \left[\tfrac{1}{\beta_1}, \tfrac{1}{\beta_0(\beta_1-1)}\right] \ \forall n \geq 0\right\}$$

$$= \Omega_{\mathbf{a}_{\beta_0,\beta_1},\mathbf{b}_{\beta_0,\beta_1}} \setminus \{0,1\}^*\{\mathbf{a}_{\beta_0,\beta_1}, \mathbf{b}_{\beta_0,\beta_1}\}$$

$\mathbf{a}_{\beta_0,\beta_1}$ quasi-greedy (β_0,β_1)-expansion of $\frac{1}{\beta_1} = \pi_{\beta_0,\beta_1}(1\overline{0})$,

$\mathbf{b}_{\beta_0,\beta_1}$ quasi-lazy (β_0,β_1)-expansion of $\frac{1}{\beta_0(\beta_1-1)} = \pi_{\beta_0,\beta_1}(0\overline{1})$

Unique β-expansions, $\beta \in (1,2]$ ($\beta_0 = \beta_1$)

$$\pi_\beta(i_1 i_2 \cdots) := \sum_{k=1}^{\infty} \frac{i_k}{\beta^k}$$

$$\begin{aligned} U_\beta &:= \big\{\mathbf{u} \in \{0,1\}^\infty \;:\; \pi_\beta(\mathbf{u}) \neq \pi_\beta(\mathbf{v}) \text{ for all } \mathbf{v} \neq \mathbf{u}\big\} \\ &= \big\{\mathbf{u} \in \{0,1\}^\infty \;:\; \pi_\beta(\Sigma^n \mathbf{u}) \notin \big[\tfrac{1}{\beta}, \tfrac{1}{\beta(\beta-1)}\big] \ \forall n \geq 0\big\} \\ &= \Omega_{\mathbf{a}_\beta, \mathbf{b}_\beta} \setminus \{0,1\}^* \{\mathbf{a}_\beta, \mathbf{b}_\beta\} \end{aligned}$$

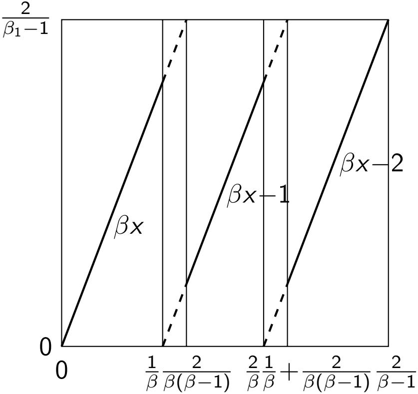
$\mathbf{a}_\beta$ quasi-greedy β-expansion of $\frac{1}{\beta} = \pi_\beta(1\overline{0})$,
$\mathbf{b}_\beta$ quasi-lazy β-expansion of $\frac{1}{\beta(\beta-1)} = \pi_\beta(0\overline{1})$
$\mathbf{b}_\beta = F(\mathbf{a}_\beta) \qquad (F(0) = 1,\ F(1) = 0)$
$s(\mathbf{b}_\beta) = F(s(\mathbf{a}_\beta)) \quad (F(L) = R,\ F(M) = M,\ F(R) = L)$

$$U_\beta = \{\overline{0}, \overline{1}\} \iff \beta \leq \tfrac{1+\sqrt{5}}{2}$$

$$U_\beta \text{ zero entropy} \iff \mathbf{a}_\beta \leq \overline{M}(\overline{0}) \quad \text{(Thue–Morse sequence)}$$

Unique β-expansions with digits $0, 1, 2$

$$U_\beta(A) := \{\mathbf{u} \in A^\infty : \pi_\beta(\mathbf{u}) \neq \pi_\beta(\mathbf{v}) \text{ for all } \mathbf{v} \neq \mathbf{u}\}$$



$$U_\beta(\{0,1,2\}) \neq \{\overline{0}, \overline{2}\} \iff \beta > 2$$

Allouche '83, Komornik–Loreti '02, Allouche–Frougny '09:

$$U_\beta(\{0,1,2\}) \text{ zero entropy } \iff \mathbf{a}_\beta \leq \iota(0\overline{N}(\overline{2})) = 0210201210120210\cdots$$

$$N: \ 0 \mapsto 01_2 \quad 1_0 \mapsto 02 \quad 1_2 \mapsto 20 \quad 2 \mapsto 21_0$$

$$\iota: \ 0 \mapsto 0 \quad 1_0 \mapsto 1 \quad 1_2 \mapsto 1 \quad 2 \mapsto 2$$

Two holes

$$\Omega_{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}} := \left\{\mathbf{u} \in \{0,1,2\}^\infty \,:\, \Sigma^n(\mathbf{u}) \notin (\mathbf{a},\mathbf{b}) \cup (\mathbf{c},\mathbf{d}) \ \ \forall n \geq 0\right\}$$
$$(\mathbf{a} \in 0\{0,1,2\}^\infty,\ \mathbf{b} \in 1\{0,1,2\}^\infty,\ \mathbf{c} \in 1\{0,1,2\}^\infty,\ \mathbf{d} \in 2\{0,1,2\}^\infty)$$

$$U_\beta(\{0,1,2\}) = \Omega_{\mathbf{a}_\beta,\mathbf{b}_\beta,\mathbf{c}_\beta,\mathbf{d}_\beta} \setminus \{0,1,2\}^*\{\mathbf{a}_\beta,\mathbf{b}_\beta,\mathbf{c}_\beta,\mathbf{d}_\beta\}$$
$$\Sigma\mathbf{a}_\beta = \Sigma\mathbf{c}_\beta,\ \Sigma\mathbf{b}_\beta = \Sigma\mathbf{d}_\beta,\ \mathbf{d}_\beta = F(\mathbf{a}_\beta),\ \mathbf{c}_\beta = F(\mathbf{b}_\beta)$$
$$(F(0) = 2,\ F(1) = 1,\ F(2) = 0)$$

$$\Omega_{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}} \text{ zero entropy},\ \Sigma\mathbf{a} = \Sigma\mathbf{c},\ \Sigma\mathbf{b} = \Sigma\mathbf{d},\ \mathbf{d} = F(\mathbf{a}),\ \mathbf{c} = F(\mathbf{b})$$
$$\iff \mathbf{a} \leq \iota(0\overline{N}(\overline{2}))$$

$$\begin{array}{rlrlrl} \iota: & 0 \mapsto 0 & N': & 0 \mapsto 02 & N'': & 0 \mapsto 01_0 \\ & 1_0 \mapsto 1 & & 1_0 \mapsto 1_0 0 & & 1_0 \mapsto 1_0 1_2 \\ & 1_2 \mapsto 1 & & 1_2 \mapsto 1_2 2 & & 1_2 \mapsto 1_2 1_0 \\ & 2 \mapsto 2 & & 2 \mapsto 20 & & 2 \mapsto 21_2 \end{array}$$

$$\iota(0\overline{N}(\overline{2})) = \iota\overline{N'N''}(\overline{0}) = 021020121012021 0\cdots$$