Controlling the Chaos of Gibbs Measures

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15/11/2022, Rencontre IZES

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Chaotic Gibbs Measures

General Framework

Subshifts of Finite Type and Forbidden Patterns

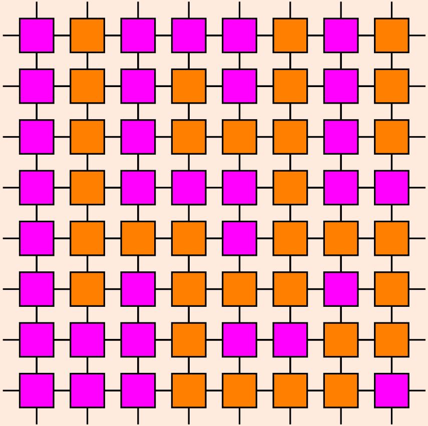


Figure 1: Example of configuration,

- Group $G = \mathbb{Z}^2$ with 2 generators.
- Alphabet $\mathcal{A} = \{■, ■\}$.

Subshifts of Finite Type and Forbidden Patterns

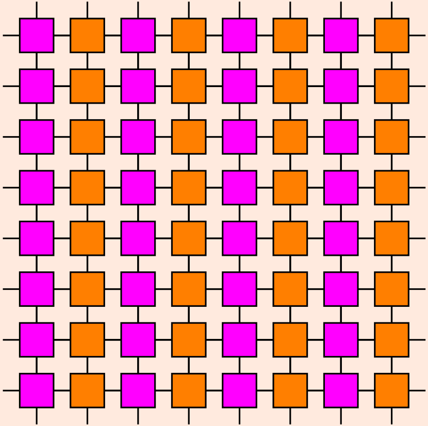
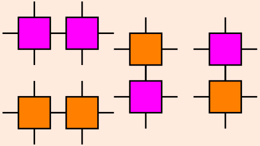


Figure 1: Example of configuration, without occurrences of the forbidden patterns.

- Group $G = \mathbb{Z}^2$ with 2 generators.
- Alphabet $\mathcal{A} = \{$, $\}$.
- Finite set of forbidden patterns $\mathcal{F}$:



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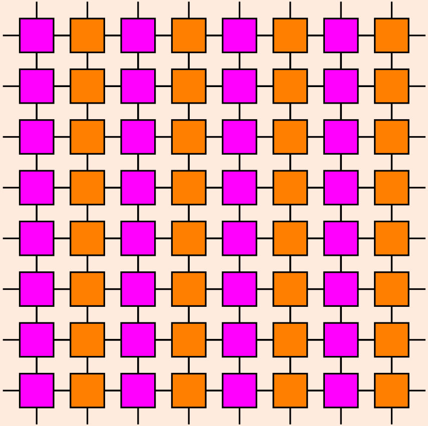

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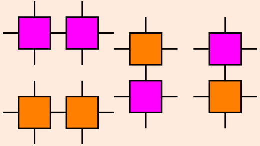

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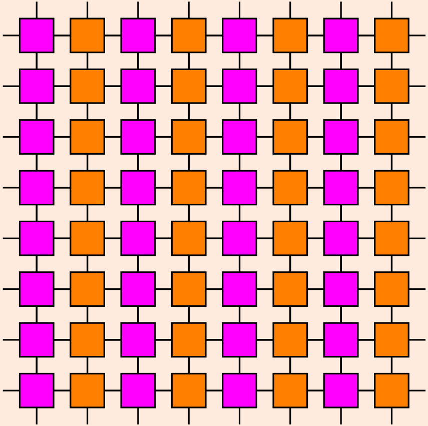

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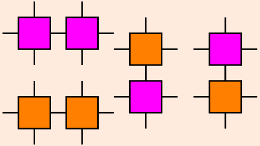

- The SFT is the space $\Omega_{\mathcal{F}} \subset \mathcal{A}^G$ of such configurations.
- Denote $\mathcal{M}_{\mathcal{F}}$ the space of translational-invariant measures.

Gibbs Measures

Local viewpoint on a finite phase space Ω:

- Energy $E : \Omega \to \mathbb{R}^+$,
- Inverse Temperature $\beta \in \mathbb{R}^+$,
- Gibbs Measure $\mu_\beta(\omega) := \frac{1}{Z} \exp(-\beta E(\omega))$.

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Global translational-invariant viewpoint on $\Omega_\mathcal{A} = \mathcal{A}^{\mathbb{Z}^2}$:

- Finite range potential $f : \mathcal{A}^{I_r} \to \mathbb{R}^+$ with $I_r = [\![-r, r]\!]^2$,
- Pressure function $P(\mu, \beta) = h(\mu) - \beta\mu(f)$ for $\mu \in \mathcal{M}_\mathcal{A}$,
- Gibbs measures $\mathcal{G}(\beta) = \operatorname{argmax}_\mu P(\mu, \beta)$,
- Ground states $\mathcal{G}(\infty) := \operatorname{Acc}(\mathcal{G}(\beta), \beta \to \infty)$.

A Link Between Worlds

$\mathcal{F}$ (forbidden patterns)	$\to$	$\Omega_{\mathcal{F}}$ (tilings)	$\to$	$\mathcal{M}_{\mathcal{F}}$ (measures)
f (potential)	$\to$	$\mathcal{G}(\beta)$ (Gibbs measures)	$\to$	$\mathcal{G}(\infty)$ (ground states)

A Link Between Worlds

$$
\begin{array}{ccccc}
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Chaotic Gibbs Measures

Chaos Ensues

Weak Chaoticity

Definition

The model is said to be *weakly chaotic* if:

$$\text{For any choice of } \mu_\beta \in \mathcal{G}(\beta),\ \#\mathrm{Acc}\left(\mu_\beta, \beta \to \infty\right) \geq 2.$$

Theorem [Chazottes and Shinoda, 2020, Barbieri et al., 2022]

There is a 2D set of forbidden patterns $\mathcal{F}$ that induces a weakly chaotic system.

Strong Chaoticity

Definition

Assume $\#\mathcal{G}(\infty) \geq 2$. The model is said to be *strongly chaotic* if:

$$\text{For any choice of } \mu_\beta \in \mathcal{G}(\beta),\ \mathrm{Acc}\left(\mu_\beta, \beta \to \infty\right) = \mathcal{G}(\infty).$$

Theorem [Gayral, Sablik and Taati, 2022]

There is a 2D set of forbidden patterns $\mathcal{F}$ that induces a strongly chaotic system.

What's more, we can obtain $\mathcal{G}(\infty) = f(X)$,
with X being any Π_2-computable connected subset of $\mathcal{M}\left(\{0,1\}^{\mathbb{N}}\right)$,
and $f : \mathcal{M}\left(\{0,1\}^{\mathbb{N}}\right) \to \mathcal{M}_{\mathcal{F}}$ an appropriate convex bijection, an affine homeomorphism.

Bounding the Chaotic Complexity

Computational Complexity of Uncountable Sets

Let (X, d) a metric space with a countable dense basis $\mathcal{B}$.

Definition

Let $Y \subset X$ be a closed set and $\mathcal{N}(Y) := \{(x, r) \in \mathcal{B} \times \mathbb{Q}^{+*}, \overline{B(x, r)} \cap Y \neq \emptyset\}$.

The set Y is said to be Π_k-computable *iff* the countable set $\mathcal{N}(Y)$ is, *i.e.* there is a computable φ such that:

$$(x, r) \in \mathcal{N}(Y) \Leftrightarrow \forall y_1, \exists y_2, \forall y_3, \dots, \varphi(x, r, y_1, \dots, y_k)$$

Here, for σ-invariant measures $\mathcal{M}_{\mathcal{A}}$ with the weak-* topology, we use the periodic measures $\widehat{\delta_w}$, with $w \in \mathcal{A}^{[\![0,n-1]\!]^d}$, as a basis $\mathcal{B}$.

Equivalent Characterisation of Π_2

Theorem [Hellouin de Menibus and Sablik, 2018]

There is a characterisation of Π_2-computable sets through accumulation points:

$$Y \in \Pi_2 \quad \Leftrightarrow \quad Y = Acc\left(x_n, n \to \infty\right) \text{ with } (x_n) \in \mathcal{B}^{\mathbb{N}} \text{ computable.}$$

$$Y \in \Pi_2 \text{ and connected} \quad \Leftrightarrow \quad Y = Acc\left(\left[x_n, x_{n+1}\right], n \to \infty\right) \text{ with } (x_n) \in \mathcal{B}^{\mathbb{N}} \text{ computable.}$$

Computable Approximations of Gibbs Measures

Let f be a finite range potential, and $P(\mu, \beta) = h(\mu) - \beta\mu(f)$ the corresponding pressure.

Lemma

The maximal pressure function $\beta \mapsto \max_\mu P(\mu, \beta)$ is computable.

There is a computable family of finite sets $\left(G^n_{a,b}\right)_{n\in\mathbb{N}, a\leq b\in\mathbb{Q}^{+*}}$ such that:

- *$G^n_{a,b}$ is a set of dyadic measures on $[\![0, n-1]\!]^d$ with near-maximal pressure (repeated independently in all directions and averaged),*
- $\bigcup_{a\leq\beta\leq b} \mathcal{G}(\beta) = Acc\left(G^n_{a,b}, n \to \infty\right).$

General Upper Bound

Theorem

We have $\overline{B(x,r)} \cap \mathcal{G}(\infty) \neq \emptyset$ *iff:*

$$\forall \varepsilon \in \mathbb{Q}^{+*}, \forall a \in \mathbb{N}, \quad \exists b \in \mathbb{N}_{>a}, \exists n_0 \in \mathbb{N}, \quad \forall n \in \mathbb{N}_{>n_0},$$
$$\exists y \in G^n_{b-1,b+1}, \overline{B(y,\varepsilon)} \cap \overline{B(x,r)} \neq \emptyset.$$

Consequently, we have a Π_3 upper bound on the complexity of $\mathcal{G}(\infty)$.

Strongly Chaotic Upper Bound

Assume now that the system is strongly chaotic.

Lemma

There is a sequence of $\beta_k \to \infty$ *such that* diam $(\mathcal{G}(\beta_k)) \to 0$.

Without loss of generality, we have rational parameters $\beta_k \in \mathbb{Q}$.

Theorem

We have $\overline{B(x,r)} \cap \mathcal{G}(\infty) \neq \emptyset$ *iff:*

$$\forall \varepsilon \in \mathbb{Q}^{+*}, \forall \beta_0 \in \mathbb{Q}^{+*}, \qquad \exists \beta \in \mathbb{Q}^{+*}_{>\beta_0}, \exists y \in \mathcal{B},$$
$$\mathcal{G}(\beta) \subset B(y,\varepsilon) \text{ and } B(y,\varepsilon) \cap \overline{B(x,r)} \neq \emptyset.$$

Consequently, we have a Π_2 upper bound on the complexity of $\mathcal{G}(\infty)$.

Perspectives

The difference in the general and the chaotic bound suggests a link between:

- the dynamic notion of chaotic systems of Gibbs measures,
- the computational complexity of the zero-temperature limit set itself.

Ultimately, we would like to prove the optimality of both bounds.

A first step in this direction is to implement connected Π_2 sets as strongly chaotic limits.

Controlling Markers Distribution

General Idea of the Weak Chaoticity

We have two measures $\lambda \neq \lambda'$ s.t. $d(\lambda, \lambda') \geq r$ and:

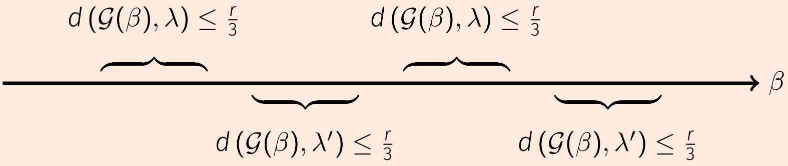

Figure 2: Alternating between incompatible behaviours on non-overlapping intervals.

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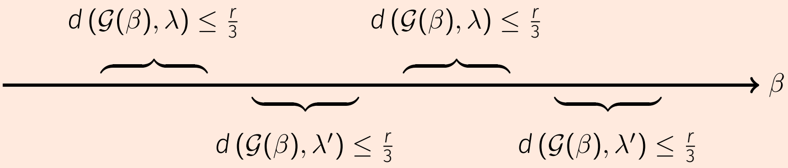

Figure 2: Alternating between incompatible behaviours on non-overlapping intervals.

Thus $\mathrm{Acc}(\mu_\beta)$ must intersect the disjoint neighbourhoods of both λ and λ'.

Moving Onto Strong Chaoticity

We want (λ_n) and $\varepsilon_n \to 0$ s.t.:

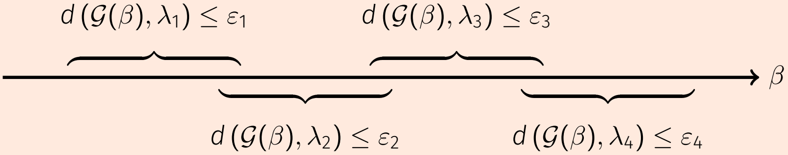

Figure 3: Phasing through similar behaviours on overlapping intervals.

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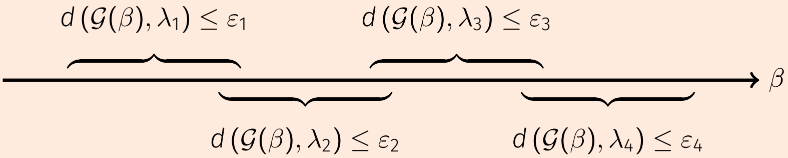

Figure 3: Phasing through similar behaviours on overlapping intervals.

Thus $\mathrm{Acc}\left(\mu_\beta\right) = \mathcal{G}(\infty) = \mathrm{Acc}\left(\lambda_n, n \to \infty\right)$.

Control of Markers on a Temperature Interval

Definition

We consider a *marker* set $Q \subset \mathcal{A}^B$ (with $B = I_k$), made of non-overlapping patterns, such that any locally admissible tiling $\omega \in \mathcal{A}^{I_{(2+\rho)k}}$ must contain a marker somewhere.

Theorem [Adapted from Chazottes-Hochman]

Denote G_n the admissible tilings of I_n, and μ_Q the cond. measure on Q of $\mu|_{\mathcal{A}^B}$.
We have constants C, C' s.t. for any marker set Q and $\varepsilon, \kappa > 0$, if:

$$\frac{\log(\#G_n)}{\#I_n} \geq (1-\kappa)\frac{\log(\#Q)}{\#B} \quad \text{and} \quad C\frac{\#B}{\varepsilon} \leq \beta \leq C' n\varepsilon,$$

then for any $\mu \in \mathcal{G}(\beta)$:

$$\mu(Q \text{ covers } 0) = 1 - O(\varepsilon + \rho) \quad \text{and} \quad H(\mu_Q) \geq (1-2\kappa)\log(\#Q) - H(\kappa) - O(\varepsilon + \rho).$$

Building an Appropriate Structure (*aka* LEGO for Grownups)

The Robinson Tiling(s)

Folkloric Robinson Tiling (Non-Overlapping Markers)

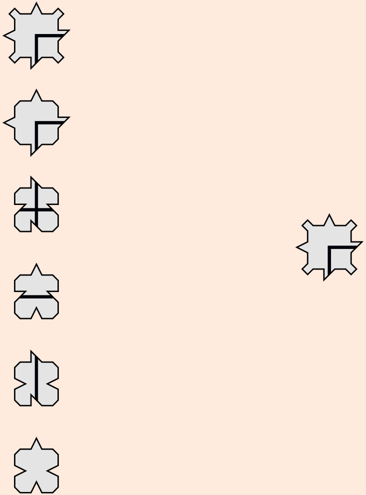

Figure 4: Hierarchical structure of the Robinson tiling.

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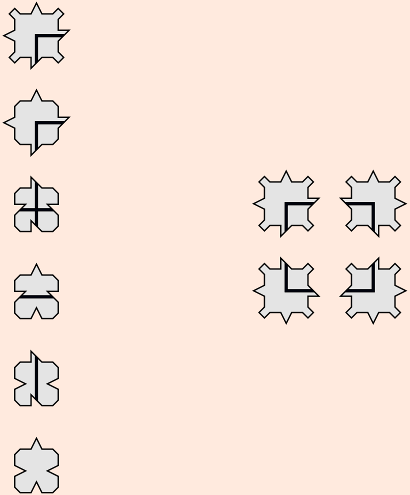

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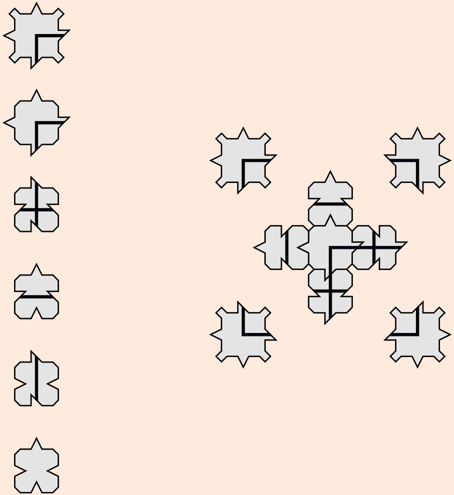

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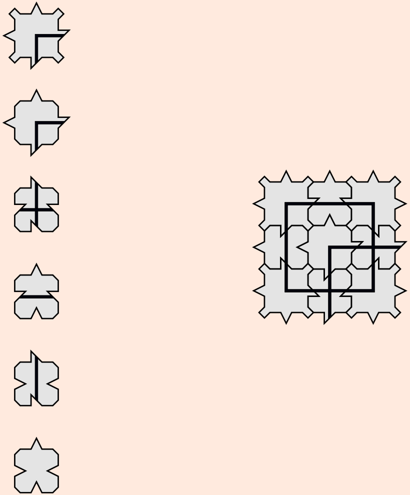

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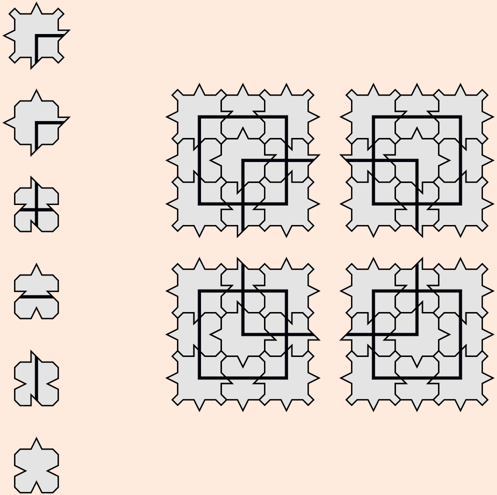

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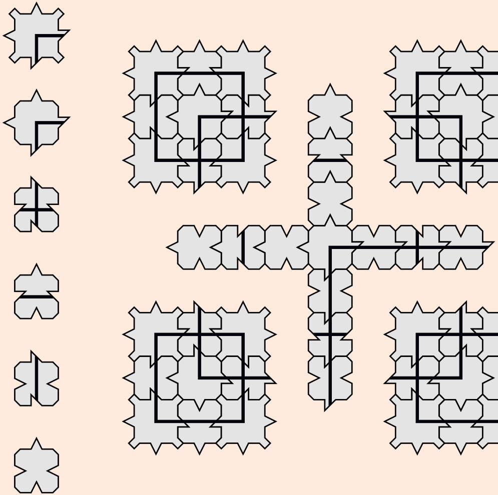

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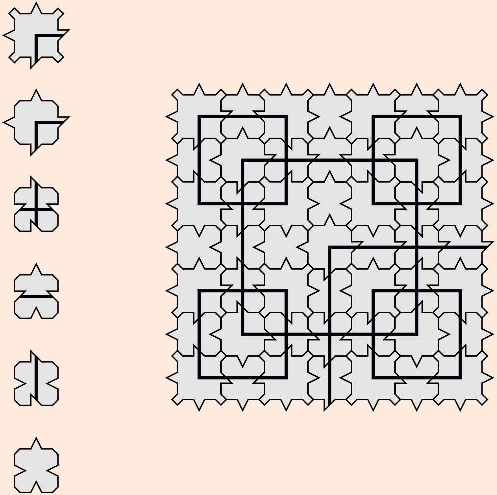

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Enhanced Robinson Tiling (Markers with Reconstruction)

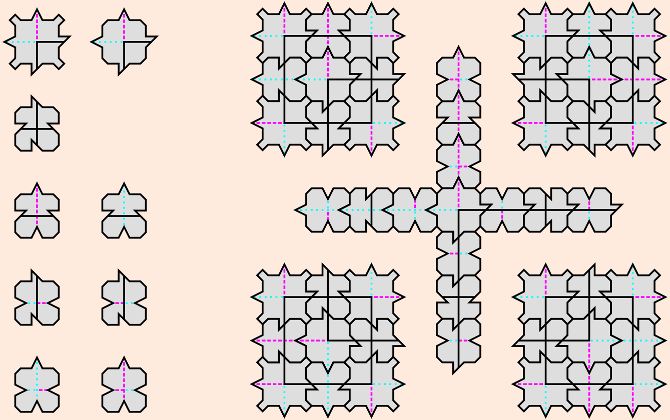


Figure 5: A Robinson variant, with strengthened local rules.

Two-Coloured Robinson for Turing Machines (Markers with Computation Area)

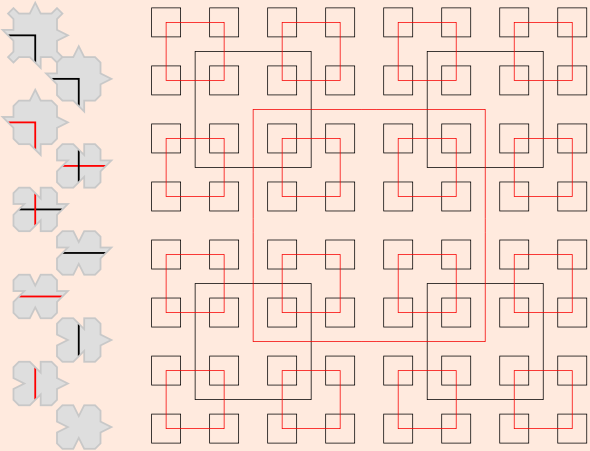
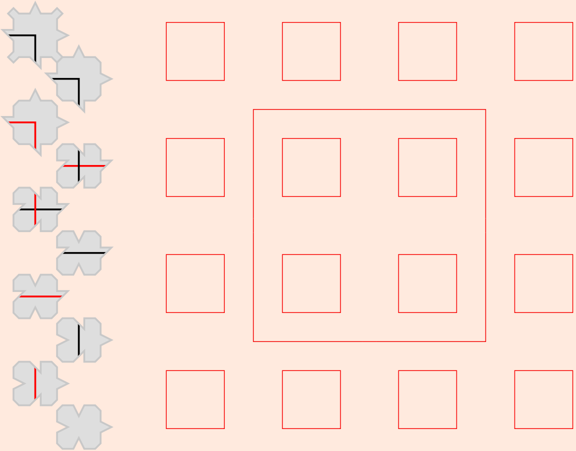


Figure 6: Alternating Red-Black structure,

Two-Coloured Robinson for Turing Machines (Markers with Computation Area)



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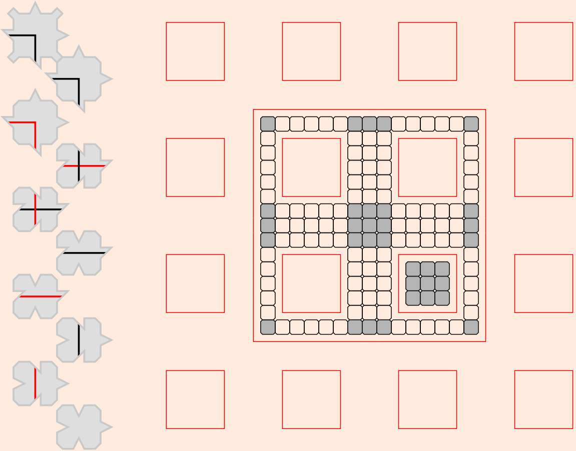

Figure 6: Alternating Red-Black structure, with a sparse computation area.

Structural Properties of the Base Layer

- The n-macro-tile has a length $l_n = 2^n - 1$.
- The n-macro-tiles are non-overlapping.
- Any locally admissible window of length $2l_n + c$ contains a n-macro-tile. (adapted from [Gayral and Sablik, 2021, Proposition 7.7])
- The N-th Red square occurs in a $(2N + 1)$-macro-tile.
- The N-th Red square has a length $4^N + 1$.
- The N-th Red square has a sparse computing area of horizon $2^N + 1$.

Building an Appropriate Structure (*aka* LEGO for Grownups)

Structure for Entropy Control

Hot and Frozen Areas

Red squares may be Blocking, with a Hot exterior and Frozen core.
The rest must locally synchronise on Hot or Frozen.

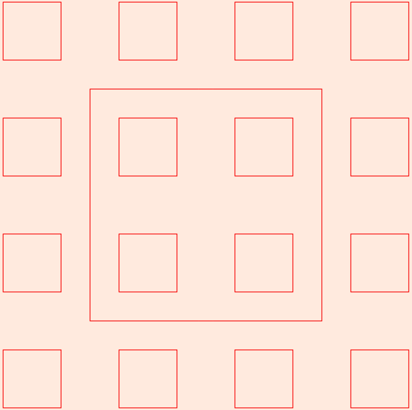

Figure 7: Admissible configurations for Hot and Frozen squares.

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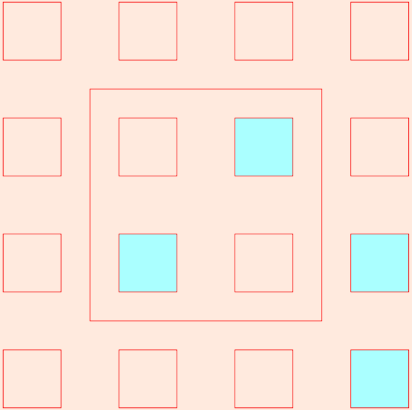

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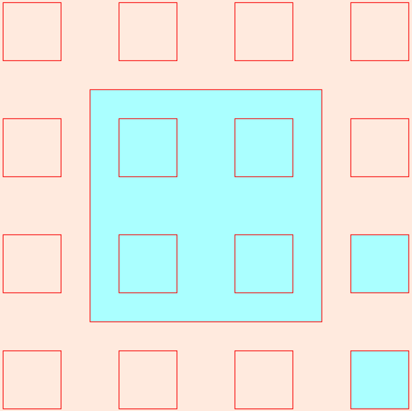

Figure 7: Admissible configurations for Hot and Frozen squares.

Blockable Scales

We (can) unary encode N as an input for computations in the N-th Red square.

We check whether $N = 3^k$. If not, the Red square *cannot* be Blocking.

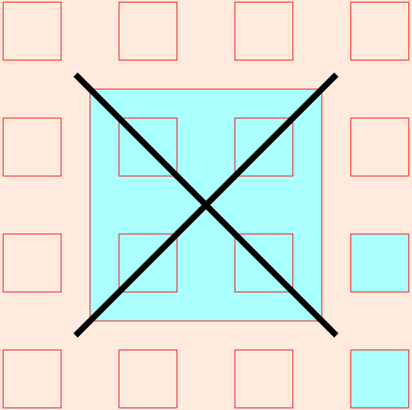

Figure 8: The 2nd scale of Red squares cannot be Blocking.

Scales for the Marker Sets

- Q_k the set of $\left(2 \times 3^k + 1\right)$-macro-tiles (Robinson layer) on the window B_k, the 3^k-th scale of locally admissible tiles with Red squares.

- A $(k+1)$-marker is a grid of $16^{3^k} \times 16^{3^k}$ smaller k-markers.

- These gaps in scale will allow for a control on the entropy, on the speed of convergence of $\frac{\log(\#Q_k)}{\#B_k} \to 0$.

Odometer

We implement an odometer in k-markers, that cycles with period $t_k = 2^{\lfloor \log_2(\lfloor \log_2(k) \rfloor) \rfloor}$, so that Red squares are Blocking once for each cycle.

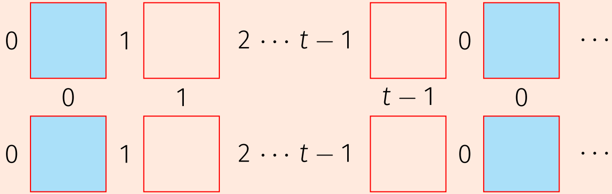

Figure 9: The repartition of Frozen squares is forced by the odometer.

The Red square of a $(k+1)$-marker initialises k-markers at 0 on one side.

Repartition of Frozen Tiles

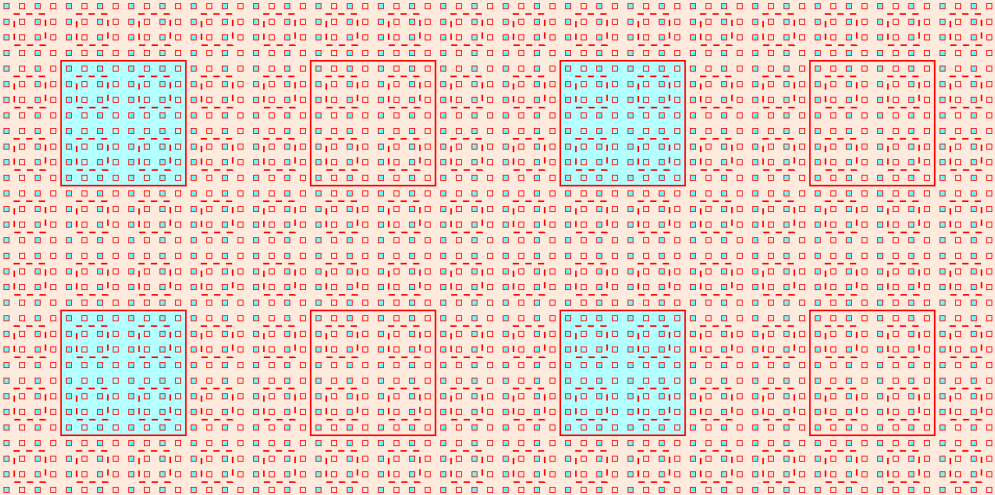

Figure 10: Approximation of a *Total Perspective Vortex.*
(One 2-marker would be a 4096×4096 grid of such 1-markers.)

Repartition of Frozen Tiles

The average scale of Blocking squares in a k-marker goes to ∞ as $k \to \infty$.

Lemma

Fix a microscopic scale m.

The proportion of non-Frozen m-markers in a k-marker is of order:

$$\prod_{j=m+1}^{k} \left(1 - \frac{1}{4t_j}\right) \xrightarrow[k\to\infty]{} 0.$$

Thus, generically, a tiling $\omega \in \Omega_{\mathcal{F}}$ is totally Frozen.

Encoding Words

Encode a letter on Red lines so that:

- Blocking and Hot squares are labelled 0,
- Frozen squares are labelled ± 1,
- Neighbouring Frozen squares synchronise their bit.

Generically, a (Frozen) tiling $\omega \in \Omega_{\mathcal{F}}$ encodes a sequence of bits in $\{\pm 1\}^{\mathbb{N}}$.

Counting Markers

Let $Q_k = Q_k^H \sqcup Q_k^B \sqcup Q_k^F$ depending on whether the Red square is Hot, Blocking or Frozen.

Proposition

We have:

- $\#Q_k^H \approx C_k^{16^{3^k}}$ with $2^{4^{-k}} \leq C_k \leq 2$,
- $\#Q_k^B \approx \left(\#Q_k^H\right)^{\frac{3}{4}}$,
- $\#Q_k^F \leq C^{4^{3^k}}$ for some $C > 1$.

Thus, $\#Q_k \approx \#Q_k^H$.

Using this result along with the bound on $H\left(\mu_{Q_k}\right)$,
we conclude that μ_{Q_k} is close to the uniform distribution on Q_k^H.

Forcing a Distribution on Words?

Now that we have a well-behaved structure,
we want to run a Turing machine in each Blocking square,
to force a distribution on the encoded words.

This will easily give us strongly chaotic examples.

The next step will be to study carefully the kind of Turing machines we can use,
to conclude on the kind of limit sets $\mathcal{G}(\infty)$ we can have.

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THE END OF PRESENTATION

ONE MORE SLIDE:

Thank you.